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Strong Asymmetry and the Mode, Median, and Mean Inequality

J. Gama[a], C. E. Suárez[b*]

[a]UFMG, Av. Pres. Antônio Carlos, 6627, Belo Horizonte, Brazil
[b]IMPA, Estrada Dona Castorina 110, Rio de Janeiro, Brazil
[a]Email: camplise@impa.br
[b]Email: gamatorres1000@gmail.com

Abstract

This paper introduces the consoled strong asymmetry and variation coefficient as generalizations of the standard measures of Pearson skewness coefficient and variance. Using these new measures, the concept of strong asymmetry is introduced. We prove that the median-mean inequality is valid for that kind of distribution, but even in that case, there is no relation between mode and median. A property similar to first-order stochastic dominance is proved for the variation coefficient. We also discuss the implications of that concept into economics.

Keywords: Skewness; Mode-Median-Mean inequality; Measures of dispersion.

1. Introduction

There is a false belief that for every unimodal and positive skewed random variable (that is $E(X-EX)^3 \geq 0$), the inequality of $mode \leq median \leq mean$ holds. Several authors have shown that it is invalid, and they have found conditions under which that inequality is valid. Reference [1] showed a list of counterexamples for all the possible combinations of orders between the three measures. Reference [11] found a condition under which $median \leq mean$, but this condition is not related to Skewness. Instead it is a relation in the cumulative distribution function. Reference [2] have extensive work about the class of distributions of fixed variance and the possible sorts in the three central measures. Inspired by advances in economics such as Behavioral Economics and specifically Prospect Theory, we have extended the classical measure of asymmetry based on the polynomial x^3 to any odd and non-decreasing function. Using this new measure, we define a class of distributions named strong asymmetric, which have the property that its corresponding measure of skewness relative to any increasing odd and convex function in R^+ is positive. The inequality $median - mean$ and $mode - mean$ can be established for this new class of functions. Similarly, a generalization of the variance is shown and several properties related to stochastic dominance.

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* Corresponding author.

2. Strong asymmetry

Similar to some notions introduced in [10,9] we make the next definitions.

Definition 1: A function $f: R \to R$ is called odd-concave if it is odd and concave in R^+ . Analogously, an odd-convex function is odd and convex in R^+

Definition 2: Let X be a random variable and f a continuous, increasing, and odd function; we define

$$Sk_f(X) = E\left[f(X - E(X))\right] \quad (1)$$

as the Skewness coefficient of X relative to the function f, and

$$Va_f(X) = E\left[f(|X - E(X)|)\right] \quad (2)$$

as the total variation coefficient of x relative to the function f.

Remark 3: Suppose that f is a continuous, increasing, and odd function and X is a real random variable with distribution F, such that $E[X] = 0$. The skewness of X relative to f is

$$\begin{aligned} Sk_f(X) &= E\left[f(X)\right] = \int_{-\infty}^{\infty} f(x)dF(x) \\ &= \int_{-\infty}^{0} f(x)dF(x) + \int_{0}^{\infty} f(x)dF(x) \end{aligned} \quad (3)$$

So, if $Sk_f(X)$ is positive,

$$\int_0^{\infty} f(x)dF(x) < \int_0^{\infty} f(x)dF(x) \quad (4)$$

The right side of the distribution is under the weights given by f. (the positive side) is stronger than the left side. Suppose additionally that f is an odd-concave function. Therefore. $\frac{f(x)}{x}$ is a decreasing function, and $Sk_f(X)$ is considerably more sensitive to values close to zero than to values far from it, then we might expect that $F'(x) > F'(-x)$ for values near to 0, i.e, intuitively, the distribution is right skewed when it is restricted to values near to the reference point. However, since $F'(x) > F'(-x)$ for values near to 0 we might have that, for values far from 0, $F'(x) < F'(-x)$, which suggests that the distribution is left-skewed. Thus, a positive f skewness coefficient with f odd-concave suggests that the distribution is left-skewed, and that $Sk_g(F)$ might be negative if g is a odd-convex function. On another side, if X is a symmetric random variable $\forall x, F'(x) = F'(-x)\ so, Sk_f(X) = 0$.

Remark 4: Some properties of the Skewness coefficient relative to f are:

1. if F has a symmetric distribution then $Sk_f(F) = 0$ for every odd function f. If for every odd and increasing function f, $Sk_f(X) = 0$ then X is symmetric.
2. If $f(x) = x^3$ then $Sk_f(x)$ correspond with the third moment around the mean.
3. If $f(x) = \eta x$ then for all x, $Sk_f(x) = 0$.

Example 5: Consider F the lottery given by $(-L, 1-p; G, p)$ with $G, L > 0$, that is, there is a probability p to gain G and probability $1-p$ of loss L. Suppose that $E(L) = 0$ so $p = \frac{L}{G+L}$ and $1 - p = \frac{G}{G+L}$. Let f strictly odd-convex

$$Sk_f(F) = E(f(F)) = (1-p)f(-L) + pf(G) = -(1-p)f(L) + pf(G) \quad (5)$$

then $Sk_f(F) > 0$ iff

$$\frac{L}{G+L}f(G) > \frac{G}{G+L}f(L) \implies Lf(G) > Gf(L) \implies \frac{f(G)}{G} > \frac{f(L)}{L} \quad (6)$$

Because $f(0) = 0$ and f is strictly odd-convex, the last relation is valid iff $G > L$. Analogously if f is strictly odd-concave $Sk_f(F) > 0$ if $G < L$.

The last example inspires the following definition:

Definition 6: A random variable X is strongly asymmetric to the right if for all f increasing and odd-convex, $Sk_f(X) \geq 0$. Analogously X is strongly asymmetric to the left if for all f increasing and odd-convex, $Sk_f(X) \leq 0$.

Example 7: Let X a random variable with exponential distribution and density function is given by

$$\varphi(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases} \quad (7)$$

This distribution is strongly asymmetric to the right. Let f be a continuous and odd-convex function then

$$\begin{aligned} Sk_f(X) = E\left[f(X-1)\right] &= \int_0^\infty f(x-1)e^{-x}dx \\ &= \frac{1}{e}\left[-\int_0^1 f(y)(e^y - e^{-y})dy + \int_1^\infty f(y)e^{-y}dy\right] \end{aligned} \quad (8)$$

Because of the convexity of f, for each $0 \leq y \leq 1$, $f(y) \leq yf(1)$, we have that

$$\int_0^1 f(y)(e^y - e^{-y})dy \leq \int_0^1 yf(1)(e^y - e^{-y})dy = \frac{2}{e}f(1) \quad (9)$$

On other side for each $y > 1$, $f(y) > f'(1)(y-1) + f(1)$, and we have that

$$\begin{aligned}\int_1^\infty f(y)e^{-y}dy &\geq \int_1^\infty (f'(1)(y-1) + f(1))\, e^{-y}dy \\ &\geq \tfrac{1}{e}(f'(1) + f(1)) \\ &\geq \tfrac{2}{e}f(1).\end{aligned}$$

Therefore, we have that.

$$Sk_f(X) = \frac{1}{e}\left[-\int_0^1 f(y)e^y dy + \int_0^\infty f(y)e^{-y}dy\right] \geq 0. \quad (10)$$

In particular, note that if f is strictly odd-convex, the inequalities above are strict.

Motivated by Example 7, we have the following result.

Theorem 8: Let $X: (-\infty, \infty) \to R$ a random variable with finite mean and density function given by $\varphi(x)$. Suppose that exist $a > E(X)$ such that

1.$\forall x \in [E[X], a), \varphi(E[X] + x) - \varphi(E[X] - x) \leq 0$

2.$\forall x \in [a, \infty), \varphi(E[X] + x) - \varphi(E[X] - x) \geq 0$

Then, the random variable is$X$ is strongly asymmetric to the right.

Proof. Suppose without loss of generality that $E(X) = 0$. Let be $g(x) = \varphi(x) - \varphi(-x)$,we want to prove that for every odd-convex function f

$$\int_{-\infty}^{\infty} f(x)\varphi(x)dx \geq 0 \quad (11)$$

Due to the parity of f,, we have

$$\int_{-\infty}^{\infty} f(x)\varphi(x)dx = \int_0^a f(x)g(x)dx + \int_a^\infty f(x)g(x)dx \quad (12)$$

Since $f(x)$ is convex in R^+ $f(x) \leq \frac{f(a)}{a}x$, for all $0 \leq x \leq a$ and $f(x) \geq f'(a)(x-a) + f(a)$, for all $x > a$ therefore.

$$f(x)g(x) \geq \frac{f(a)}{a} x g(x), \text{ for all } 0 \leq x \leq a$$

$$f(x)g(x) \geq (f'(a)(x-a) + f(a))g(x), \text{ for all } x > a \quad (13)$$

We conclude that:

$$\int_0^a f(x)g(x)dx + \int_a^\infty f(x)g(x)dx \geq$$

$$\frac{f(a)}{a}\int_0^a xg(x)dx + f'(a)\int_a^\infty xg(x)dx + (f(a) - af'(a))\int_a^\infty g(x)dx =$$

$$-\frac{f(a)}{a}\int_a^\infty xg(x)dx + f'(a)\int_a^\infty xg(x)dx + (f(a) - af'(a))\int_a^\infty g(x)dx =$$

$$\left[f'(a) - \frac{f(a)}{a}\right]\left[\int_a^\infty (x-a)g(x)dx\right] \geq 0. \quad (14)$$

Remark 9: Because of Theorem 8, distributions such as the gamma family are strongly asymmetric to the right.

Theorem 10: A random variable X is strongly asymmetric to the right if and only if for all g increasing and odd-concave, $Sk_g(X) \leq 0$.

Proof. First, suppose that there exists a constant $\eta > 0$ such that for all $x \in R^+$, $\eta x \geq g(x)|_{R^+}$, then the function $h(x) = \eta x - g(x)$ is increasing, odd and convex, so $Sk_h(x) \geq 0$.

$$0 \leq Sk_h(x) = E\left[\eta(x - E(x)) - g(x - E(x))\right] = \eta E(x - E(x)) - Sk_g(x)$$

$$Sk_g(x) \leq 0.$$

Now suppose that there is no $\eta>0$ satisfying that, for all $x \in R^+$, $\eta x \geq g(x)|_{R^+}$, define

$$g_n(x) = \begin{cases} nx & \text{if } |x| < r_n \\ g(x) & \text{if } |x| \geq r_n \end{cases} \quad (15)$$

where $r_n > 0$ satisfies that $nr_n = g(r_n)$, such r_n exist because of the concavity of g. We have that $r_n \to 0$ and that $g_n(x)$ converges uniformly to $g(x)$. Applying the first part we have that $Sk_{g_n}(X) \leq 0$ and by convergence $Sk_g(X) \leq 0$. Conversely consider $f(x)$ odd concave and increasing function. We will assume that f is differentiable (an analogous analysis is made using subgradient) and that $E(X) = 0$. For any $\alpha > 0$ consider the function

$$g_\alpha(x) = \begin{cases} f'(\alpha)x - f(x) & |x| \leq \alpha \\ \text{sign}(x)\left(f'(\alpha)\alpha - f(\alpha)\right) & |x| > \alpha \end{cases} \quad (16)$$

Then g_α is oan dd-concave and increasing function. By hypothesis, if F is the distribution function of the random variable X,

$$0 \geq \int g_\alpha(x)dF(x) = \int_{|x|\leq\alpha} f'(\alpha)x - f(x)dF(x) + \int_{|x|>\alpha} \text{sign}(x)\left(f'(\alpha)\alpha - f(\alpha)\right)dF(x)$$

adding $\int_{|x|>\alpha} f'(\alpha)xdF(x) - \int_{|x|>\alpha} f'(\alpha)xdF(x) + \int_{|x|>\alpha} f(x)dF(x) - \int_{|x|>\alpha} f(x)dF(x)$, and organizing the terms we obtain that for all $\alpha > 0$,

$$\int f(x)dF(x) \geq \int_{|x|>\alpha} f(x) - f'(\alpha)x + \text{sign}(x)\left(f'(\alpha)\alpha - f(\alpha)\right)dF(x)$$

And finally, we have that.

$$\lim_{\alpha\to\infty} \int_{x>\alpha} f(x) - f(\alpha) - f'(\alpha)(x-\alpha)dF(x) = 0$$

so $\int f(x)dF(x) \geq 0$

2.1. Strong asymmetry

A common conceptual mistake is to say that if a distribution \$F\$ is unimodal and the Pearson skewness coefficient is positive, then it is true that $mode(F) \leq median(F) \leq mean(F)$. \cite{abadir2005mean} showed several examples of the violations of each one of the inequalities. When a strongly asymmetric is considered, it is possible to show that $median(F) \leq mean(F)$

Theorem 11. If X is a random variable strongly asymmetric to the right, then the median of X is less or equal to the mean of X.

Proof. Let $h(x) = 1_{x>0} - 1_{x<0}$ and $g_n(x)$ a sequence of functions converging to $h(x)$ such that each $g_n(x)$ is odd concave and increasing (note that there is no sequence of odd-convex functions converging to $h(x)$), then $Sk_{g_n}(x) \leq 0$. Let F the distribution function of the random variable X then,

$$\int g_n(X - EX)dF(x) \leq 0 \implies$$

$$-\int_{-\infty}^{EX} g_n(EX - X)dF(x) + \int_{EX}^{\infty} g_n(EX - X)dF(x) \leq 0 \implies$$

$$\int_{EX}^{\infty} g_n(EX - X)dF(x) \leq \int_{-\infty}^{EX} g_n(EX - X)dF(x) \quad (17)$$

Because $|g_n(x)| \leq |h(x)| = 1$, we can apply the dominated convergence theorem. Finally, we conclude that the median must be lesser or equal to $E[X]$.

Theorem 12. If X is a uni-modal random variable strongly asymmetric to the right, then the mode of X is less or equal to its mean.

Proof. We will proceed by contradiction supposing that $E[X] = 0$ and $Mode(X) = a > 0$. Let be $h_x(x)$ the correspondent density function and define $g(x) = h_x(x) - h_x(-x)$. We have that $h(x)$ is increasing close to 0, then there exist ε > 0 such that $g(x) > 0$ for all $x \in (0, \varepsilon]$. Because $\int_0^\infty xg(x)dx = 0$ there exist the minimum positive root of $r \neq 0$ of $g(x)$. Let $f(x) = (x - r)1_{[r,\infty)}$, clearly f is non decreasing, positive and convex.

$$\int_0^\infty g(x)f(x)dx = \int_r^\infty (x - r)g(x)dx \leq \int_r^\infty xg(x)dx = -\int_0^r xg(x)dx < 0 \quad (18)$$

which contradicts the hypothesis of strong asymmetry to the right.

Remark 13. Even in the case of strong asymmetry, there is no relation between mode and median. In fact for (see figure 1)

$$h(x) = \begin{cases} 0.13x^{-4.7} & x > 1 \\ -0.085x + 0.215 & 1 \leq x \leq -1 \\ 0.0891x + 0.3891 & -4.3666 < x < -1 \end{cases} \quad (19)$$

We have that the distribution satisfies the criteria of theorem 8 for strong asymmetry and

$$median = -1.0167 < mode = -1 < mean = -0.9983$$

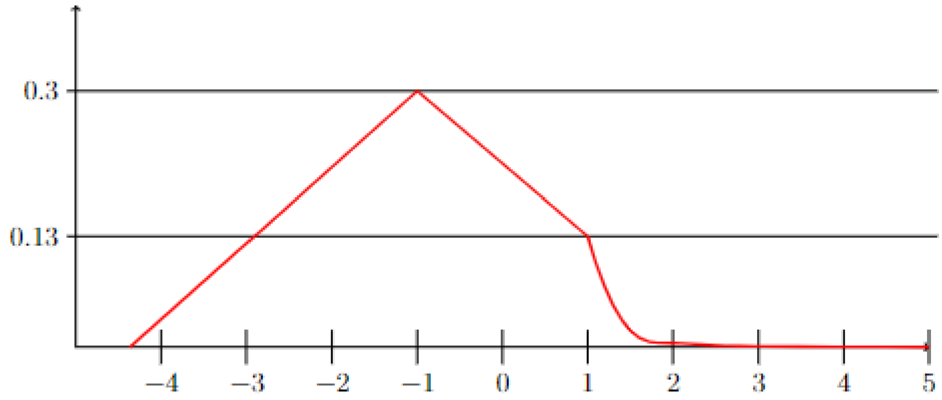

Figure 1: Strong distribution to the right with median < mode < mean.

In [11] is proved that the class of distributions for which the median is located between the mode and the mean is characterized by the relation $\forall u > 0, F(m-u) + F(m+u) \leq 1$, but this condition is nor related with Skewness but instead a relation in the cumulative distribution function.

2.2. Variation coefficient.

As in the case of skewness, if the agent is risk-averse, they perceive more the risk for marginal changes close to the reference point than those who are further away. In that sense, there is a difference with the CAPM model since the variance gives greater weight to points far from the average. This would correspond to agents whose distortion function is convex. Some properties of the variation coefficient are: $Va_f(x) \geq 0$, and $Va_f(x) = 0$ if and only if x is constant.

1. If $f(x) = x^2$ then $Va_f(x)$ corresponds with the variance.
2. If $f(x) = x$ then $Va_f(x)$ is corresponds with the mean absolute deviation MAD.
3. By Jensen inequality, $Va_f(x) \geq f\big(MAD(x)\big)$ if f is odd-convex, and $Va_f(x) \leq f\big(MAD(x)\big)$ if f is odd-concave.
4. $Va_f(x) \geq \left|Sk_f(x)\right|$
5. If f is odd-concave, $Va_f(x)$ satisfies triangular inequality, i.e. $Va_f(x+y) \leq Va_f(x) + Va_f(y)$. On other side $Va_f(\alpha x) \leq |a| Va_f(x)$ if $\alpha \geq 1$, and $Va_f(\alpha x) \geq |a| Va_f(x)$ if $\alpha < 1$.

Remark 15: Let be X and Y two random variables with distribution functions F and G respectively, then for all f odd and increasing function, $Va_f(X) \geq Va_f(Y)$ if and only if for all $r > 0$, $P(|X - EX| < r) \leq P(|Y - EY| < r)$. That corresponds to the "dispersion order" notion introduced in [2].

2.3. Relation with behavioral economics.

Consider agents that behave according to Prospect Theory and their reference point is endogenous, similar to [3,8,5]. The value function is generally considered concave for gains and convex for losses, and even more [7],

[6] described how the negative side is in some way the reflection of the positive one. The losses, in general, correspond to the reflection of the gains with respect to the straight line $y = -x$ multiplied by a constant λ denominated loss aversion coefficient, whose value has been determined to be between 1.5 and 5. If $f(x)$ represent the concave value function for gains then the value function μ is equivalent to

$$\mu(x) = \min\big(f(x), \lambda f(x)\big) = \frac{1}{2}(f(x) + \lambda f(x) - |f(x) - \lambda f(x)|) \quad (20)$$

If we consider that outcomes have a distribution F and the endogenous reference point is given by $E(X)$,

$$E(\mu(X - E(X)) = \frac{\lambda+1}{2}\int_X f(c-r)dF(c) - \frac{\lambda-1}{2}\int_X f(|c-r|)dF(c) =$$

$$\frac{(\lambda+1)}{2}Sk_f(F) - \frac{(\lambda-1)}{2}Va_f(F) \quad (21)$$

From this representation and using the properties listed previously in this paper, it is possible to measure the impact of agents as described in real economies.

3. Conclusions

In this paper, we showed that the standard measure of skewness could be extended for the class of odd and increasing functions. A more robust notion of asymmetry can be defined and is consistent with all the expected skewness properties.

We prove that for strong asymmetric distributions, the inequalities $median \leq mean\$ and \$mode \leq mean$ are valid, but even in that case, there is no relation between the mode and the median.

Finally, there is a strong relationship between the skewness and variation coefficients defined in this paper with works in regret, disappointment, and Prospect theory such as [3,8,5].

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