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An Alternate Method to Find Area of Triangle from its Vertices alone in a Plane

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Abstract

Since the time of ancient Greeks, mathematicians have been interested in finding the areas of basic plane regions. From among the most basic plane regions are triangles. Triangular shapes are used in different areas of engineering, especially in the design and analysis of trusses and in finding moments of inertia in mechanics and to find triangulation and trilateration in surveying. The centroid of a triangular region can be also expressed in terms of the coordinates of vertices of triangles. Literatures have shown that there are various methods of finding an area of a triangle. From among these are using the length of the base and height of a triangle, using the length of two sides and the sine of included angle or using Heron`s formula which uses the length of all sides of the triangle. We can also find the area of a triangle if the coordinates of the vertices are given using either vectors or determinants. However, vectors and determinants are concepts of higher mathematics. In this article, we present a method that enables us to find the area of a triangle without the knowledge of vectors or determinants, provided that the coordinates of the vertices of a triangle were given. We use analytic approach to derive the formula. The method use only elementary arithmetic. To this end a procedure were designed and the method were checked for accuracy using different examples. Finally, a theorem was formulated and proved.

Keywords: Area of triangle; Centroid; Vectors& determinants; Procedure of evaluation

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1. Introduction

It has been stated in [1,2,3,4,7,9,12,13,14] (elementary text books of mathematics) that the area of a triangle having base b and height h is given by

$$A = \frac{1}{2}bh \qquad (1)$$

Yet another expression for the area of a triangle is in terms of the length of its sides is credited to a Greek mathematician, who lived during the first century, Heron [7,8,10,14,15]. If we use the standard notion a, b, c for the sides of triangle PQR, then by Herons formula the area of the triangle is

$$A = \sqrt{s(s-a)(s-b)(s-c)} \qquad (2)$$

where $s = \frac{1}{2}(a+b+c)$.'

The area of a triangle can also be found from the product of the length of two sides and the sine of included angle [1], [8], [9], [10], [11], [17]

$$A = \frac{1}{2}ab\sin\theta \qquad (3)$$

where θ is the included angle.

The area of a triangle with all its vertices given can also be found by either the concept of vectors or determinants. The area A of a triangle having vertices described by the Cartesian coordinates $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$, is given by

$$A = \pm\frac{1}{2}\left\|\overrightarrow{PQ}\times\overrightarrow{PR}\right\| \qquad (4)$$

Where, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are the vectors determined by their corresponding points in a plane [16]. Or as stated in [3], using determinants, we have the formula

$$A = \pm\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \qquad (5)$$

Note that the sign $(\pm)$ stands for orientation of the vertices.

As stated in [6], the centriod of a triangle having base b and height h is located at one-third of the height from the base of the triangle and this is valid for any type of triangles.

In [5], it have been stated that the coordinates of the location of the centriod of an arbitrary triangle having vertices $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$, is

$$x_c = \frac{x_1 + x_2 + x_3}{3} \text{ and } y_c = \frac{y_1 + y_2 + y_3}{3} \qquad (6)$$

Although it is possible to find the area of a triangle from its vertices by using the concept of vectors & determinants, we come up with another way of finding the area of a triangle which uses vertical alignment of vertices and is more convenient for computing.

2. Methodology

This study is a basic research that employs mathematical exploration and experimentation. The researcher explored the concepts regarding the subject of study presented in different areas of mathematics and engineering namely geometry, analytic geometry, algebra, mechanics and surveying. In this study, the derivation of the formula for calculating the area of a triangle in a plane was done in analytic approach. The procedure was devised following the principle of problem solving and inductive approach.

A series of mathematical experiments and computations were done in order to establish the procedure of computation and the use of derived formula for different types of triangles. The results of the study were also compared with previously established rules and formulas.

In the following section, we shall present the main steps which are used in the new approach as a procedure.

3. Results and Discussion

3.1. Procedure of evaluation

Below we list the major steps of the method as a procedure that guide us in finding the area of a triangle.

Step 1 Rewrite the points of vertices vertically down in column form.

Step 2 Add the first components together and put the sum exactly under the first coordinate of the last vertex.

Step 3 Repeat step 2 for second component.

Step 4 Find the product of the results that are obtained in step 2 and 3

Step 5 Multiply the coordinates of each vertex with each other and put the product in front of each vertex in the third column.

Step 6 Find the sum of the products obtained in step 5 and change its sign & put the result under the last component under the third column.

Step 7 Draw a line from the 1st coordinate of the first vertex to the 2nd coordinate of the last vertex. Draw a line from the 1st coordinate of the second vertex to the 2nd coordinate of the first vertex. Next, draw a line from the 1st coordinate of the third vertex to the 2nd coordinate of the second vertex.

Multiply the corresponding numbers joined by the end points of the lines. Put their results under the fourth column in the 3rd, 1st and 2nd rows respectively. Find the sum of the products obtained in this step and multiply the result by -2.

Step 8 Finally, the area of the triangle is half the algebraic sum of the numbers obtained in steps 4, step 6 and step 7.

Remark1: If the calculated value is positive, this indicates that the orientation of the vertices is anti-clockwise. If the result of the calculation is negative, this indicates that the orientations of the vertices are in clockwise. In the latter case we drop the negative sign and take the positive numerical value alone for the area.

Remark2: The method is independent of the order of arrangements of the vertices of a triangle.

The formula for finding the area of a triangle from its vertices alone is given below.

Let $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$, be the Cartesian coordinates of triangle PQR. Then the above steps can be generalized and put in a compact form as shown below.

$$\begin{array}{llll} P(x_1, y_1) = & x_1y_1 & x_2y_1 \\ Q(x_2, y_2) = & x_2y_2 & x_3y_2 \\ R(x_3, y_3) = & x_3y_3 & x_1y_3 \\ \hline \end{array}$$

$$A = \frac{1}{2}\left[\left(\sum x_i\right)\left(\sum y_i\right) - \sum x_i y_i - 2\sum x_i y_j\right] \qquad (7)$$

In addition, the formula showing the relationship between i and j is

$$i = j - 2\cos\left(\frac{j2\pi}{3}\right) \text{for } 1 \le j \le 3 \qquad (8)$$

Or using the concept of permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (123)$

Thus

$$\sum_{\substack{1\le j\le 3\\ i=j-2\cos\left(\frac{j2\pi}{3}\right)}} x_i y_j = \sum_{\substack{1\le j\le 3\\ i=j-2\cos\left(\frac{j2\pi}{3}\right)}} y_j x_i = x_1y_3 + x_3y_2 + x_2y_1$$

3.2. Problems and Solutions

Sample problems and solution to illustrate the use of procedure and the derived formula are presented here.

Example 1: (Right angled triangle) Consider a triangle having vertices $P(0,0), Q(3,0)$ and $R(3,4)$.Find the area A of triangle PQR

Solution:

Rewrite the vertices as

$$\begin{array}{lcc}
P(0,\ 0) = & 0 & 0 \\
Q(3,\ 0) = & 0 & 0 \\
R(3,\ 4) = 12 & & 0 \\
\hline
\end{array}$$

$$A = \frac{1}{2}[(6)\times(4) - 12 - 2\times 0]$$

$$\begin{aligned}
\sum x_i \sum y_i &= (0+3+3)(0+0+4) \\
-\sum x_i y_i &= -((0\times 0)+(3\times 0)+(3\times 4)) = -12 \\
-2\sum x_i y_j &= -2((0\times 4)+(3\times 0)+(3\times 0)) = 0
\end{aligned}$$

Therefore,

$$A = \frac{1}{2}\left[\sum x_i \sum y_i - \sum x_i y_i - 2\sum x_i y_j\right]$$

$$A = \frac{1}{2}[24 - 12 - 0] = 6$$

The above triangle is a right angled triangle having legs of length 3 and 4. Hence the area is half the product of the length of the legs which is 6 units square.

Example 2: (30-60 degree) Consider a triangle having vertices $A(0,0), B(4\sqrt{3},0)$ and $C(4\sqrt{3},4)$.Then the area A is

Solution

Rewrite the vertices as

$$\begin{array}{lcc} A(\ 0,\quad 0),= & 0 & 0 \\ B\left(4\sqrt{3},0\right)= & 0 & 0 \\ C\left(4\sqrt{3},4\right)= & 16\sqrt{3} & 0 \\ \hline \end{array}$$

$$A=\frac{1}{2}\left[(8)\sqrt{3}(4)-16\sqrt{3}-2\times 0\right]$$

$$A=\frac{1}{2}(16)\sqrt{3}(2-1)=\frac{1}{2}(16)\sqrt{3}=8\sqrt{3}$$

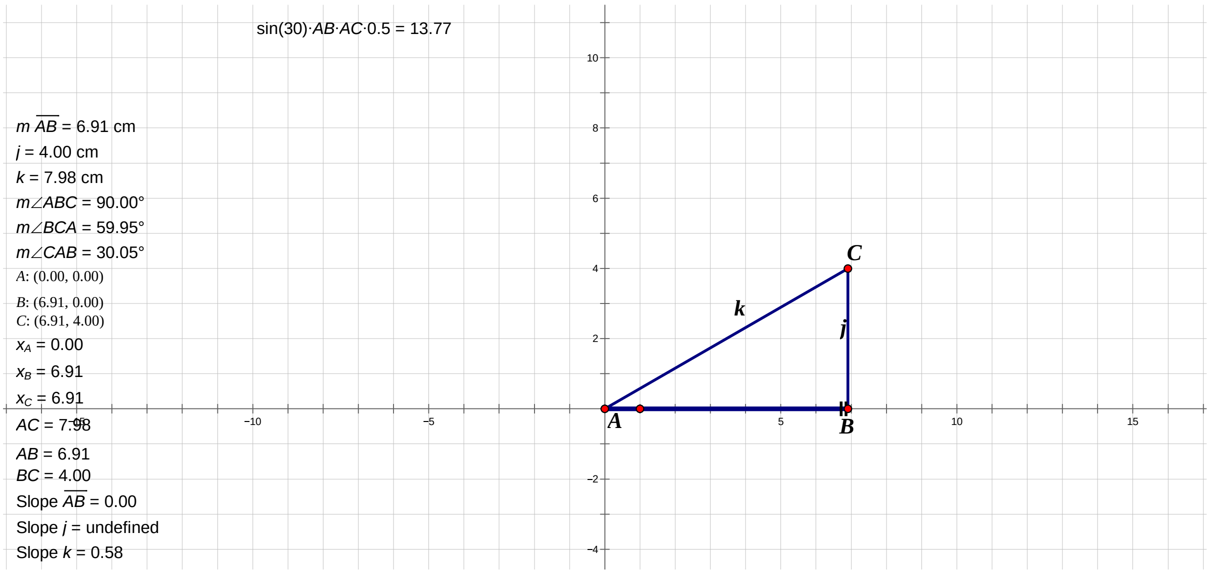

Figure1: $30^{\circ}-60^{\circ}$ degree triangle

The above triangle is a 30-60 degree right angled triangle having hypotenuse of length 8 units, legs of length $4\sqrt{3}$ and 4 units. Hence the area is half the product of the length of the adjacent leg $4\sqrt{3}$ and the hypotenuse 8 and $\sin 30$ which is $8\sqrt{3}$ units square.

Example 3: (Isosceles triangle) Consider a triangle having vertices $P(-1,4), Q(-4,2)$ and $Q(2,2)$.Then the area A is

Solution

Rewrite the vertices as

$$\begin{array}{lcc} P(-1, \quad 4), = & -4 & -16 \\ Q(-4, \quad 2) = & -8 & 4 \\ R(2, \quad 2) = & 4 & -2 \\ \hline \end{array}$$

$$A = \frac{1}{2}[(-3)(8) \quad - \quad (-8) - 2 \times (-14)]$$

$$A = \frac{1}{2}[(-24) + 36] = \frac{1}{2}(12) = 6$$

The above triangle is an isosceles triangle having height 2 units and base 6 units.

Example 4: (Obtuse triangle) Consider a triangle having vertices $P(7,4), Q(11,-2)$ and $Q(15,-2)$.Then the area A is

Solution

Rewrite the vertices as

$$\begin{array}{lcc} P(7, \ 4) = & 28 & 44 \\ Q(11,-2) = & -22 & -30 \\ R(15,-2) = & -30 & -14 \\ \hline \end{array}$$

$$A = \frac{1}{2}[(33)(0) - \quad (-24) \quad -2 \times 0]$$

$$A = \frac{1}{2}[(0) + 24 - 0] = \frac{1}{2}(24) = 12$$

The above triangle is an obtuse triangle with base and height 4 and 6 units long respectively.

Example 5: (Acute triangle) Consider a triangle having vertices $P(0,0), Q(4\sqrt{3},0)$ and $Q(4\sqrt{3},4)$.Then the area A is

Solution

Rewrite the vertices as

$$
\begin{array}{lcc}
P(0,\ \ 0)= & 0 & 0 \\
Q(0,4)= & 0 & \dfrac{28}{7} \\
R\left(\dfrac{12}{7},\dfrac{12}{7}\right)= & \dfrac{144}{49} & 0 \\
\hline
\end{array}
$$

$$A=\frac{1}{2}\left[\left(\frac{12}{7}\right)\left(\frac{40}{7}\right)-\frac{144}{49}-2\times\frac{28}{7}\right]$$

$$A=\frac{1}{2}\left(\frac{12}{7}\right)(4-8)=\frac{1}{2}\left(\frac{12}{7}\right)(-4)=-\frac{24}{7}$$

The above triangle is an acute triangle having base 4 units long and height $\frac{12}{7}$ units long.

Example 6: Show that the area of a right angled triangle with base b and height h is $A=\frac{1}{2}bh$

Solution

Let us construct a right angled triangle having base b and height h as in Fig 1 below. Without loss of generality,

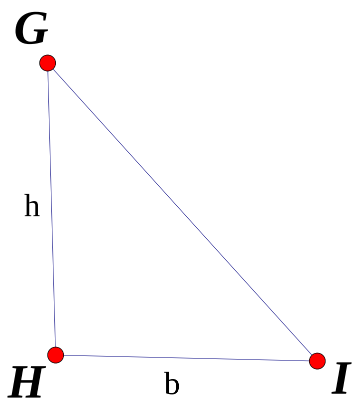

Figure 2: Right triangle

Let us take the first quadrant and the vertex of the right angle be at the origin.

The vertices of the triangle are

$$\begin{array}{ll} P(0,0), = 0 & 0 \\ Q(b,0) = 0 & 0 \\ \underline{R(0,h) = 0} & \underline{0} \end{array}$$

$$\underline{A = \frac{1}{2}[(b)(h) - 0 - 2 \times 0] = \frac{1}{2}bh}$$

Our next goal is to establish some of the basic algebraic formulas based on the procedure above. To do this, we use the concept of centroid of a triangle and area of a triangle using determinant.

Theorems

Theorem 1:

The area A of a triangle having vertices $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$, is given by

$$A = \frac{1}{2}\left[\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=1}^{3} y_i\right) - \left(\sum_{i=1}^{3} x_i y_i\right) - 2\sum_{i=1}^{3} x_i y_j\right]$$

Or for short $A = \frac{1}{2}\left[\sum x_i \sum y_i - \sum x_i y_i - 2\sum x_i y_j\right]$

Where $\sum x_i \sum y_i = (x_1 + x_2 + x_3)(y_1 + y_2 + y_3)$,

$\sum x_i y_i = x_1 y_1 + x_2 y_2 + x_3 y_3$ and

$\sum x_i y_j = x_1 y_3 + x_3 y_2 + x_2 y_1$

Proof: For the purpose of exposition, let us assume that the vertices of the triangle are oriented in ant-clockwise direction.

The centroid of the triangle is given by $x_c = \frac{x_1 + x_2 + x_3}{3}$, $y_c = \frac{y_1 + y_2 + y_3}{3}$

The product of the coordinates of the centroid is $(x_c)(y_c) = \left(\frac{x_1 + x_2 + x_3}{3}\right)\left(\frac{y_1 + y_2 + y_3}{3}\right)$

This implies that

$$9(x_c)(y_c) = (x_1 + x_2 + x_3)(y_1 + y_2 + y_3) = \sum x_i \sum y_i$$
$$= x_1y_1 + x_1y_2 + x_1y_3 + \quad x_2y_1 + x_2y_2 + x_2y_3 + \quad x_3y_1 + x_3y_2 + x_3y_3 \qquad \cdots \quad (9)$$

It has been stated in [1] that the area of the triangle is given by

$$A = \pm\frac{1}{2}\begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix} = \frac{1}{2}\left[x_2y_3 + x_3y_1 + x_1y_2 - x_1y_3 - x_2y_1 - x_3y_2\ \right] \qquad \cdots \quad (10)$$

Thus rewriting the area using (9) and (10) we have

$$A = \frac{1}{2}\begin{bmatrix} x_1y_1 + x_1y_2 + x_1y_3 + \quad x_2y_1 + x_2y_2 + x_2y_3 + \quad x_3y_1 + x_3y_2 + x_3y_3 - x_1y_1 - x_2y_2 \\ - x_3y_3 - 2x_1y_3 - 2x_2y_1 - 2x_3y_2 \end{bmatrix}$$
$$= \frac{1}{2}\left[\sum x_i \sum y_i - \sum x_i y_i - 2\sum x_i y_j\right]$$

Where $\sum x_i \sum y_i = (x_1 + x_2 + x_3)(y_1 + y_2 + y_3)$,

$\sum x_i y_i = x_1y_1 + x_2y_2 + x_3y_3$ and

$$\sum_{\substack{1\le j\le 3 \\ i = j - 2\cos\left(\frac{j2\pi}{3}\right)}} x_i y_j = \sum_{\substack{1\le j\le 3 \\ i = j - 2\cos\left(\frac{j2\pi}{3}\right)}} y_j x_i = x_1y_3 + x_3y_2 + x_2y_1$$

Theorem 2:

The area A of a parallelogram who's any three of its four vertices $P(x_1, y_1), Q(x_2, y_2)$ and $R(x_3, y_3)$, is given by

$$A = \left[\left(\sum_{i=1}^{3} x_i\right)\left(\sum_{i=1}^{3} y_i\right) - \left(\sum_{i=1}^{3} x_i y_i\right) - 2\sum_{i=1}^{3} x_i y_j\right]$$

Or for short $A = \left[\sum x_i \sum y_i - \sum x_i y_i - 2\sum x_i y_j\right]$

Where $\sum x_i \sum y_i = (x_1 + x_2 + x_3)(y_1 + y_2 + y_3)$,

$\sum x_i y_i = x_1y_1 + x_2y_2 + x_3y_3$ and

$$\sum x_i y_j = x_1 y_3 + x_3 y_2 + x_2 y_1$$

Proof: Since the diagonal of a parallelogram divides it in to two triangles having the same area, once we determine the area of one triangle, we can multiply it by two to obtain the area of the parallelogram.

4. Conclusion and Directions

Finding the area of a triangle and rectangles is very important in mathematics, mechanics and surveying. In engineering mechanics it is used to find the moment of inertia, and in the design and analysis of plane trusses. In surveying it is used in triangulation and trilateration. The technique and formula found here may be used as an alternative way of finding an area of triangles in a plane. The method is direct and can be done easily with greater accuracy and speed. The formula is derived using analytic approach.

The property of centriod of a triangle together with the method of finding its area using the method of determinants provides the present method. Despite the restriction of orientation stated in remark1 above, the formulas and procedure we were drive remain valid for all triangles and parallelograms. The method can also be used to find the area of any polygon by partitioning it into a number of triangles having common vertex. The new approach given in this article provides a more convenient and a very efficient method for calculating the area, as it uses coordinates of vertices of the triangle and simple operations of addition and multiplication of those coordinates. Hence this method is recommended for solving areas of triangles, parallelograms and quadrilaterals displayed in two dimensions.

Future Works to be done by other researchers may be finding the area of a triangle in three dimensions. i.e. extending the formula derived here for triangles in space.

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