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Shape Preserving C^2 Rational Cubic Spline Interpolation

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Abstract

In this study a piecewise rational function $S \in C^2[a,b]$ with cubic numerator and linear denominator involving two shape parameters has been developed to address the problem of constructing positivity preserving curve through positive data, monotonicity preserving curve through monotone data and convexity preserving curve through convex data within one mathematical model. A simple data dependent condition for a single shape parameter has been derived to preserve the positivity, monotonicity and convexity of respectively positive, monotone and convex data. The remaining shape parameter is left free for the user to modify the shape of positive, monotone and convex curves when needs arise. We extended the result of [1] to a piecewise rational cubic function $S \in C^2[a,b]$.

Keywords: Shape preservation; Spline interpolation; Positivity; Monotonicity; Convexity

1. Introduction

Shape preservation of a given data is an important topic in the field of data visualization. In data visualization techniques researchers convert any information into graphical views. These graphical views have great importance in many fields including engineering, military, transport, advertising, medicine, education, art, etc. Data that is used for the visualization has some hidden properties such as positive or convex [2]. Curve design plays a significant role in manufacturing different products such as ship design, car modeling, and aero plane fuselages and wings [3].

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Monthly rainfall amounts, levels of gas discharge in certain chemical reactions, progress of an irreversible process, resistance offered by an electric circuit, volume and density, etc. are some of the physical quantities which are always positive [1]. Dose–response curves and surfaces in biochemistry and pharmacology, approximations of couples and quasi-couples in statistics, empirical option pricing model in finance and approximation of potential functions in physical and chemical systems are always monotone [4]. Convexity has various applications in different disciplines including telecommunication systems designing, nonlinear programming, engineering, optimization, parameter estimation, approximation theory S. Butt, 1991 cited in [1].

A lot of research has been done on this topic. In [5] the authors used a rational cubic function in its most generalized form to preserve the shape of positive planar data. Schmidt and Heβ cited in [1] developed sufficient conditions on derivatives at the end points of an interval to assure the positivity of the cubic polynomial over the whole interval. The authors in [3] developed a C^2 piecewise rational cubic spline scheme to address the problem of constructing a positivity-preserving curve through positive data. In [2] the authors introduced C^2 rational cubic function with two families of free parameters to attain the C^2 positive curves from positive data. In [6] the authors addressed the problem of visualizing positive data by imposing C^1 continuity on rational cubic function.

Fuhr and Kallay 1992 cited in [1] used a C^1 monotone rational B-spline of degree one to preserve the shape of monotone data. The authors in [7] studied the problem of constructing a monotonicity preserving curve through monotone data using C^2 rational cubic spline. In [8] the authors visualized monotone data by using piecewise rational cubic function. The smoothness of the interpolation is C^1 continuity. Reference [9] developed an explicit representation of a C^1 piecewise rational cubic spline which can produce a monotonic interpolant to a given monotone data.

Brodlie and Butt 1991 cited in [1] preserved the shape of convex data by piecewise cubic interpolation. In any interval where convexity was lost, they divided the interval into two subintervals by inserting extra knots into that interval. The method that was presented was C^1.A C^2 rational cubic function with two families of free parameters has been introduced to attain C^2 convex curves from convex data [2].

Reference [10] surveyed the shape-preserving interpolating algorithms for 2D data. Reference [11] used cubic Hermite in a parametric form to preserve the shape of data. The step lengths were used as tension parameters to preserve the shape of planar functional data. The first-order derivatives at the knots were estimated by a tridiagonal system of equations which assured C^2 continuity at the knots. Reference [12] proposed a direct, inexpensive, constructive method for interpolating convex, monotone data using shape-preserving C^2 cubic polynomial splines [1] discussed the three important shapes within one mathematical model. They introduced a C^1 rational function with cubic numerator and cubic denominator involving four free parameters.

Positivity, monotony and convexity are very important independent shape features which are found inherited in a data, under different conditions and circumstances, in one form or the other. In computer graphics environment, a user is always in need of interpolatory schemes which preserve the shape of the data under consideration under different conditions and circumstances [1]. The problem of constructing a shape preserving curve through given data points is one of the basic problems in computer graphics, computer aided geometric design, data visualization and engineering. Classical methods, with the polynomial spline functions, show smooth and visually pleasing results but usually ignore these shape features of data and thus yield solutions exhibiting undesirable undulations or oscillations [3].

This study is a contribution towards achieving shape preserving curves for positive, monotone and convex data using one mathematical model. Several researches were focused on studying the three shape features independently using different mathematical models and methodologies. Even though, the authors in [1] tried to preserve these three shape features of data under one mathematical model and methodologies, they imposed a rational cubic interpolant $\in C^1[a,b]$. Thus, the purpose of this study was to construct a piecewise rational cubic spline function which can preserve the three important shapes within one mathematical model. In addition, this study is aimed to extend the result of [1] to a rational cubic interpolant $S \in C^2[a,b]$.

The developed scheme has the following advantageous over the other existing schemes:

- It works for both equally and unequally spaced data.
- The order of continuity attained is C^2.
- We obtained only one tridiagonal system of linear equations, for finding the values of derivative parameters.
- The schemes developed in this study is smoother and visually pleasing as compared to the other schemes with order of continuity C^1.
- The schemes developed in this study preserves these three important shape features of data within one mathematical model.

Besides shape preservation, shape control and shape design are important areas for the graphical representation of data. This paper is delimited to the area of shape preservation.

The remainder of this paper has been organized as in the following order and procedures:

- C^2 rational cubic function with two free parameters in its description has been discussed.
- We developed the conditions on a single shape parameter to preserve the positivity of a positive data.
- The conditions on a single shape parameter has been derived to preserve respectively, the monotonicity and convexity of monotone and convex data.
- We provided numerical examples and demonstrated the results visually using graphs.
- Finally, we concluded the paper and recommended.

2. Materials and Methods

Let $\{(x_i, f_i), i = 1,2,\ldots,n\}$ be a given set of data points defined over the interval $[a,b]$, where $a = x_1 < x_2 < \cdots < x_n = b$. The piecewise rational cubic function $S \in C^2[a,b]$ involving two free parameters $(u_i, v_i > 0)$, over each subinterval $I_i = [x_i, x_{i+1}]$, $i = 1,2,\ldots,n-1$, is defined by [13] as:

$$S(x) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^{i}A_i}{u_i(1-\theta)+v_i\theta} \tag{1}$$

Where $h_i = x_{i+1} - x_i$, $x_i \leq x \leq x_{i+1}$, $\theta = \frac{(x-x_i)}{h_i}$, $\theta \in [0,1]$

With C^2 interpolatory conditions:

$$S(x_i) = f_i, \qquad S(x_{i+1}) = f_{i+1},$$

$$S^{(1)}(x_i) = d_i, \qquad S^{(1)}(x_{i+1}) = d_{i+1}, \tag{2}$$

$$S^{(2)}(x_{i+}) = S^{(2)}(x_{i-}); \quad i = 2,3,\ldots,n-1;$$

Where $S^{(1)}(x)$ and $S^{(2)}(x)$ denote the first and second ordered derivatives with respect to x, the $+$ and $-$ subscripts denote the right and left derivatives respectively, d_i denote the unknown derivative values at given knots x_i that are used for the smoothness of curve.

The C^2 interpolating conditions produce the following unknown coefficients A_i , $i = 0,1,2,3$

$$A_0 = u_i f_i \qquad A_1 = f_i(2u_i + v_i) + u_i h_i d_i$$

$$A_2 = f_{i+1}(u_i + 2v_i) - v_i h_i d_{i+1} \qquad A_3 = v_i f_{i+1} \tag{3}$$

Thus equation (1) can be reduced to a rational cubic spline:

$$S(x) = S_i(x) = \frac{p_i(\theta)}{q_i(\theta)} \tag{4}$$

With,

$$p_i(\theta) = u_i f_i(1-\theta)^3 + (f_i(2u_i + v_i) + u_i h_i d_i)\theta(1-\theta)^2$$

$$+ (f_{i+1}(u_i + 2v_i) - v_i h_i d_{i+1})\theta^2(1-\theta) + v_i f_{i+1}\theta^3$$

$$q_i(\theta) = u_i(1-\theta) + v_i\theta$$

Again a C^2 interpolating conditions (2) produce the following system of linear equations for the computation of derivatives parameters $d_i(i = 2,\ldots,n-1)$.

$$\alpha_i d_{i-1} + \beta_i d_i + \gamma_i d_{i+1} = \sigma_i, \text{ for } i = 2,3,\ldots,n-1 \tag{5}$$

With;

$$\alpha_i = u_i h_i u_{i-1}$$

$\beta_i = [u_i h_i(v_{i-1} + u_{i-1}) + v_{i-1}h_{i-1}(u_i + v_i)]$

$\gamma_i = v_{i-1}h_{i-1}v_i$ and

$\sigma_i = u_i h_i(v_{i-1} + 2u_{i-1})\Delta_{i-1} + v_{i-1}h_{i-1}(u_i + 2v_i)\Delta_i$

Where $\Delta_i = (f_{i+1} - f_i)/h_i$.

Remark-1: If in each subinterval we use $u_i = v_i = 1$,then the rational cubic function (1) reduces to the standard cubic Hermite.

Remark-2: In (5) since all u_i, v_i, h_i are non-negative we see that α_i,β_i, $\gamma_i > 0$. Moreover β_i exceeds the sum of α_i and γ_i by $u_i v_{i-1}(h_i + h_{i-1})$.

$\Rightarrow \beta_i > \alpha_i + \gamma_i \Rightarrow |\beta_i| > |\alpha_i + \gamma_i|$.

Therefore eq. (5) represents a diagonally dominant system of linear algebraic equation. Thus it has a unique solution for the n-2 unknown derivative parameter and can be solved either by using the LU decomposition method or any other suitable method.

Remark-3: Since there are n unknown derivative values we need two more equations for the unique solution. Therefore, we have to impose the end conditions at end knots as:

$S^{(1)}(x_1) = d_1$ and $S^{(1)}(x_n) = d_n$ (6)

3. Results and Discussion

3.1. Positivity preserving C^2 rational cubic function

Let $\{(x_i, f_i)\}$ such that $x_i < x_{i+1}$ for all $i = 1, 2 \ldots n-1$ be the set of positive data points, we need to develop the conditions on which our interpolant produces a positive curve. Thus a rational cubic function (1) is positive if both $p_i(\theta)$ and $q_i(\theta)$ are positive. Since $q_i(\theta) > 0$ for all $u_i, v_i > 0$, we only construct the condition on a shape parameter u_i for which $p_i(\theta) > 0$.

According to the result developed by Schmidt and Heβ 1988 cited in [1] $p_i(\theta) > 0$ if

$(p_i^{(1)}(0), p_i^{(1)}(1) \in R_1 \cup R_2$ where;

$$R_1 = \left\{(a, b): a > \frac{-3p_i(0)}{h_i}, b < \frac{3p_i(1)}{h_i}\right\} \tag{7}$$

$$R_2 = \left\{\begin{matrix}(a, b): 36f_i f_{i+1}\left(a^2 + b^2 + ab - 3\Delta_i(a + b) + 3{\Delta_i}^2\right) + 4h_i(f_{i+1}a^3 - f_i b^3) \\ -h_i^2 a^2 b^2 + 3(f_{i+1}a - f_i b)(2h_i ab - 3f_{i+1}a + 3f_i b) > 0\end{matrix}\right\} \tag{8}$$

Suppose $(p_i^{(1)}(0),\ p_i^{(1)}(1)) \in R_1 \cup R_2$ then, $(p_i^{(1)}(0),\ p_i^{(1)}(1)) \in R_1$

$$\Rightarrow p_i^{(1)}(0) > \frac{-3p_i(0)}{h_i}, \qquad p_i^{(1)}(1) < \frac{3p_i(1)}{h_i}$$

Simplifying these inequalities gives

$$u_i > \frac{-v_i f_i}{2f_i + h_i d_i} \quad , \ u_i \ > \frac{v_i(h_i d_{i+1} - 2f_{i+1})}{f_{i+1}}$$

The constraints on u_i can also be derived from (8) but for simplicity we use the constraints obtained in (7) for positivity-preserving graphical results.

Theorem-1: The C^2 rational cubic spline (1) preserves positivity in each subinterval

$I_i = [x_i, x_{i+1}],\ i = 1,2,\ldots,n-1$, if the shape parameters satisfy the following conditions:

$$v_i > 0,\ u_i = w_i + max\left\{0, \frac{-v_i f_i}{2f_i + h_i d_i}, \frac{v_i(h_i d_{i+1} - 2f_{i+1})}{f_{i+1}}\right\} \ for\ some\ w_i > 0$$

The proof follows the above derivation.

Numerical example: Consider the positive data given in table-1. From Figure 1, it is clear that the cubic Hermite function has failed to preserve a positive shape of the data, whereas Figure 2 which is drawn by applying theorem-1 with $v_i = 0.25$ preserves the shape of the positive data. Table-2 demonstrates the numerical results which are computed from the developed scheme of Figure 2.

Table 1: A positive data set

i	1	2	3	4	5	6	7
x_i	1	3	5	8	10	11	12
f_i	2.50	1.00	0.05	0.03	0.14	0.20	0.22

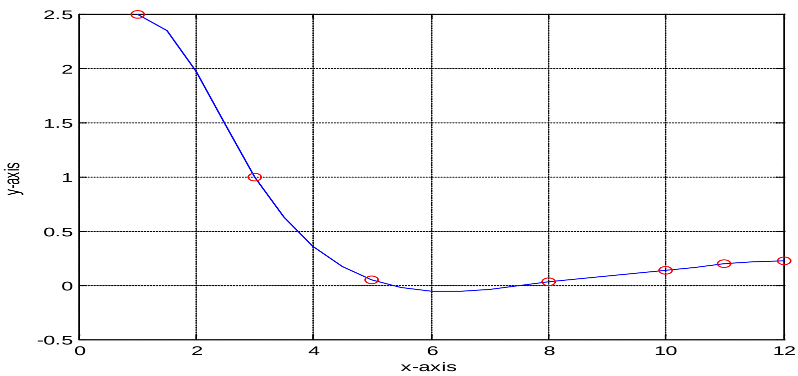

Figure 1: Cubic Hermite function for positive data

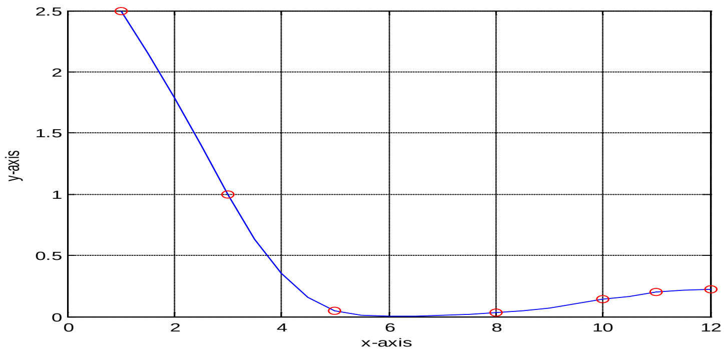

Figure 2: Positivity preserving C2 rational cubic function with $\boldsymbol{v_i = 0.25}$

Table 2: Numerical results of Figure **2**

$\boldsymbol{i}$	$\boldsymbol{d_i}$	Δ_i	$\boldsymbol{u_i}$	$\boldsymbol{v_i}$
1	0	-0.7500	0.0010	0.25
2	-0.8192	-0.4750	0.0010	0.25
3	-0.1291	-0.0067	0.0445	0.25
4	0.0302	0.0550	0.0010	0.25
5	0.0799	0.0600	0.0010	0.25
6	0.0400	0.0200	0.0010	0.25
7	0	$-$	$-$	$-$

3.2. Monotonicity preserving C^2 rational cubic function

Let $\{(x_i, f_i)\ i = 1, 2 \ldots n-1\}$ $i = 1, 2 \ldots n-1$ be the set of monotone increasing data points, such that $f_1 \leq f_2 \leq \ldots \leq f_n$, or $\Delta_i \geq 0$. Suppose $d_i \geq 0$ then the given piecewise rational cubic spline function (1) is monotone increasing in each subinterval $I_i = [x_i, x_{i+1}]$, if $S^{(1)}(x) \geq 0$. Differentiating (1) ones with respect to x we get:

$$S^{(1)}(x) = \frac{r_i(\theta)}{h_i[q_i(\theta)]^2} \tag{9}$$

With $r_i(\theta) = (1-\theta)^3 B_0 + \theta(1-\theta)^2 B_1 + \theta^2(1-\theta)B_2 + \theta^3 B_3$

Where

$$B_0 = u_i^2 h_i d_i$$

$$B_1 = h_i[u_i^2(2\Delta_i - d_i) + 2u_i v_i(2\Delta_i - d_{i+1})]$$

$B_2 = h_i[v_i^2(2\Delta_i - d_{i+1}) + 2u_i v_i(2\Delta_i - d_i)]$

$B_3 = v_i^2 h_i d_{i+1}$

Now $S^{(1)}(x) \geq 0$ provided that $B_0, B_1, B_2, B_3 \geq 0$. From our hypothesis $B_0, B_3 \geq 0$ is obvious.

Both B_1, B_2 are non-negative if $u_i \geq \frac{v_i(d_{i+1} - 2\Delta_i)}{2\Delta_i - d_i}$

Theorem-2: The C^2 rational cubic spline (1) preserves monotonicity in each subinterval

$I_i = [x_i, x_{i+1}],\ i = 1, 2, \ldots, n - 1$, if the shape parameters satisfy the following conditions:

$v_i > 0,\ u_i = k_i + max\left\{0, \frac{v_i(d_{i+1} - 2\Delta_i)}{2\Delta_i - d_i}\right\}\ for\ some\ k_i > 0$

The proof follows the above derivation.

Numerical example

A monotone data set given in table-2 below which shows the world population from the year $1000 - 2011$ in billions is borrowed from [7]. Figure 3, which is drawn by Cubic Hermite function, does not preserve the monotonicity of this data. On the other hand, Figure 4 which is drawn by monotonicity preserving C^2 rational cubic function stated under theorem−2 preserves the shape of this monotone data everywhere. Table−4 demonstrates the numerical results which are computed from the developed scheme of Figure 4

Table 3: A monotone data set

i	1	2	3	4	5	6	7	8	9	10
Years (x_i)	1000	1250	1500	1920	1960	1980	1990	2000	2005	2011
Population (f_i)	0.31	0.40	0.50	1.86	3.02	4.44	5.27	6.06	6.45	7.02

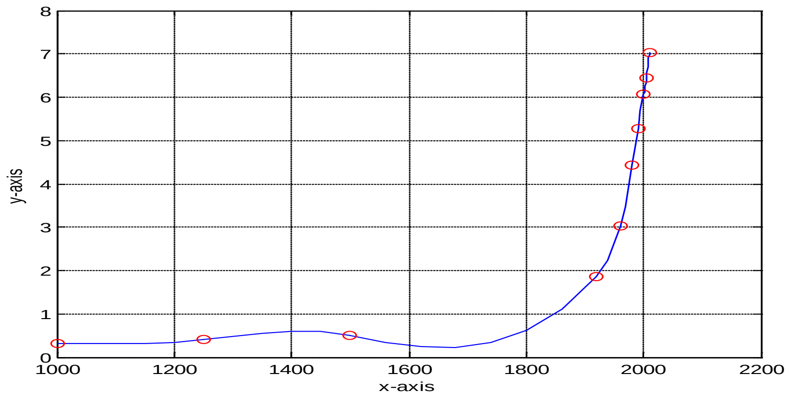

Figure 3: Cubic Hermite function for monotone data

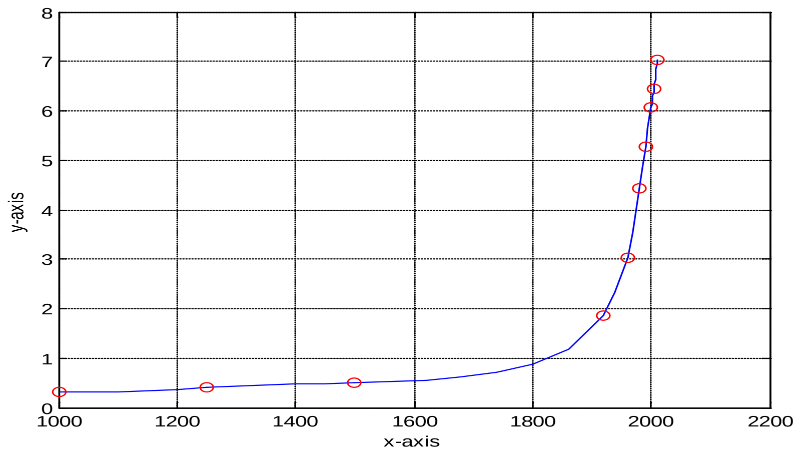

Figure 4: Monotonicity preserving C^2 rational cubic function with $\boldsymbol{v_i = 0.25}$

Table 4: Numerical results of Figure **4**

i	d_i	Δ_i	u_i	v_i
1	0.0000	0.0004	1.5	0.25
2	0.0006	0.0004	1.5	0.25
3	0.0001	0.0032	1.5	0.25
4	0.0191	0.0290	1.5	0.25
5	0.0451	0.0710	1.5	0.25
6	0.0909	0.0830	1.5	0.25
7	0.0766	0.0790	1.5	0.25
8	0.0802	0.0780	1.5	0.25
9	0.0801	0.0950	1.5	0.25
10	0.0000	–	–	–

3.3. Convexity preserving C^2 rational cubic function

In this section, we discuss the problem of convexity preserving C^2 rational cubic spline. Let $\{(x_i, f_i)\}$ such that $\Delta_i < \Delta_{i+1}$ for all $i = 1, 2 \ldots n-1$ be the set of convex data points.

Suppose $d_i < \Delta_i < d_{i+1}$ $for\ i = 2, \ldots n-2$ then, for the given piecewise rational cubic spline function (1) to produce a convex curve, we need $S^{(2)}(x) \geq 0$.Differentiating (1) twice with respect to x we get:

$$S^{(2)}(x) = \frac{(1-\theta)^3 C_0 + \theta(1-\theta)^2 C_1 + \theta^2(1-\theta) C_2 + \theta^3 C_3}{h_i[u_i(1-\theta) + v_i\theta]^3} \qquad (10)$$

Where

$C_0 = 2u_i^3(\Delta_i - d_i) + 2u_i^2 v_i(\Delta_i - d_i) - 2u_i^2 v_i(d_{i+1} - \Delta_i)$

$C_1 = 6u_i^2 v_i(\Delta_i - d_i)$

$C_2 = 6u_i v_i^2(d_{i+1} - \Delta_i)$

$C_3 = 2v_i^3(d_{i+1} - \Delta_i) + 2u_i v_i^2(d_i - \Delta_i) + 2u_i v_i^2(d_{i+1} - \Delta_i)$

From (10), $S^{(2)}(x) \geq 0$ if $C_i \geq 0$ *for all* $i = 0,1,2,3$. Since $d_i < \Delta_i < d_{i+1}$ by assumption, we have

$C_1 > 0$ *and* $C_2 > 0$.

To get $C_0 \geq 0$ we must have:

$2u_i^3(\Delta_i - d_i) \geq 2u_i^2 v_i(d_{i+1} - \Delta_i)$,

This gives $u_i \geq \frac{v_i(d_{i+1}-\Delta_i)}{\Delta_i - d_i}$ (11)

To get $C_3 \geq 0$ we must have:

$2v_i^3(d_{i+1} - \Delta_i) \geq -2u_i v_i^2(d_i - \Delta_i)$

This gives $u_i \leq \frac{v_i(d_{i+1}-\Delta_i)}{\Delta_i - d_i}$ (12)

From (11) and (12) we get $u_i = \frac{v_i(d_{i+1}-\Delta_i)}{\Delta_i - d_i}$

Therefore all the above discussion yields to the following theorem.

Theorem-3: The C^2 rational cubic spline (1) preserves convexity in each subinterval

$I_i = [x_i, x_{i+1}]$, $i = 1, 2, \ldots, n-1$, if the shape parameters satisfy the following conditions:

$$v_i > 0, u_i = m_i + \frac{v_i(d_{i+1} - \Delta_i)}{\Delta_i - d_i} \quad \textit{for some } m_i > 0$$

The proof follows the above derivation.

Numerical example:

Table 5 below which show a convex data set is the modification of a non-convex data set given under table 4 of [1]. The authors wrongly considered the data as if it was convex. Figure 5, which is drawn by Cubic Hermite function, does not preserve the convexity of this data. On the other hand, Figure 6 which is drawn by convexity preserving C^2 rational cubic function given under theorem−3 preserves the shape of this convex data everywhere. Table−6 demonstrates the numerical results which are computed from the developed scheme of Figure 6.

Table 5: A convex data set

i	1	2	3	4	5	6	7
x_i	-4	-3.5	-2	0	2	3.5	4
f_i	5	3	-1.5	-1.75	-1.5	3	5

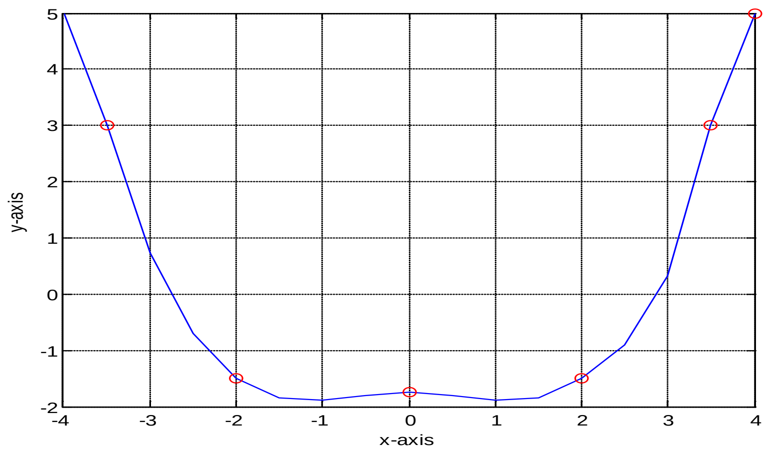

Figure 5: Cubic Hermite function for convex data

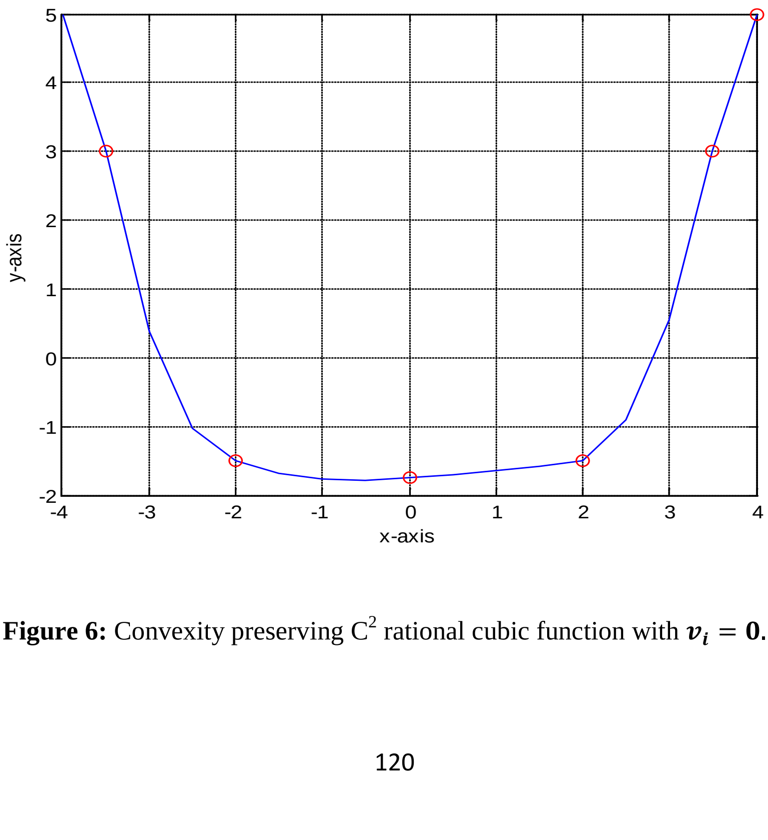

Figure 6: Convexity preserving C^2 rational cubic function with $\boldsymbol{v_i = 0.1}$

Table 6: Numerical results of Figure **6**

i	d_i	Δ_i	u_i	v_i
1	0.0000	−4.0000	0.25	0.1
2	−6.4895	−3.0000	0.25	0.1
3	−0.4362	−0.1250	0.25	0.1
4	0.0998	0.1250	1.5	0.1
5	0.3971	3.0000	0.25	0.1
6	4.6556	4.0000	0.8	0.1
7	0.0000	−	−	−

4. Conclusion

In order to assure smooth visualization of shaped data, C^2 rational cubic splines are constructed. A piecewise rational cubic spline function $S \in C^2[a,b]$ with cubic numerator and linear denominator involving two free shape parameters has been developed to address the problem of constructing positivity preserving curve through positive data, monotonicity preserving curve through monotone data and convexity preserving curve through convex data. A simple data dependent condition for a single shape parameter is derived to preserve the positivity, monotonicity and convexity of respectively positive, monotone and convex data. The remaining shape parameter is left free for the user to modify the shape of positive, monotone and convex curves when the needs arise. Each of the positivity, monotony and convexity preserving schemes has been supported with practical demonstrations on various examples of data. The developed schemes are applicable to such problems in which only data points are known. We developed a tridiagonal system of linear algebraic equations with the help of C^2 interpolating conditions and this enabled us to find the unknown derivative values.

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References

[1] M. Sarfraz, M.Z. Hussain & M. Hussain. "Shape preserving curve interpolation." *International Journal of Computer Mathematics*, vol.89, pp.35-53, 2012.

[2] M.Z. Hussain, M. Sarfraz and T.S. Shaikh. "Shape preserving rational cubic spline for positive and convex data." *Egyptian Informatics Journal*,vol. 12, pp.231–236, 2011.

[3] M.Abbas, A.Abd.Majid , Hj. Awang, Md.N. and J. Md. Ali. "Positivity-preserving C^2 rational cubic spline Interpolation." *ScienceAsia*, vol.39,pp.208–213,2012.

[4] G. Beliakov. "Monotonicity Preserving Approximation Of Multivariate Scattered Data." *BIT Numer. Math.*, vol.45, pp. 653–677,Aug.2005.

[5] M.Z. Hussain and M.Sarfraz. "Positivity-preserving interpolation of positive data by rational cubics." *J. Comput. Appl. Math,. vol.218, pp. 446–458,2008.*

[6] M.Abbas, A.Abd. Majid, and J. Md. Ali. "A Rational Spline for Preserving the Shape of Positive Data." *International Journal of Computer and Electrical Engineering*, vol.5,2013.

[7] M.Abbas, A.Abd. Majid, and J. Md. Ali. "Monotonicity-preserving C^2 rational cubic spline for monotone data." *Applied Mathematics and Computation*, vol.219, pp.2885–2895 ,2012.

[8] M.Abbas, A. Abd. Majid, Hj. Awang, Md.N. and J.Md. Ali . "Monotonicity Preserving Interpolation using Rational Spline." *Proceedings of the International multi Conference of Engineers and Computer scientists I,* 2011.

[9] M.Tian. "Monotonicity-Preserving Piecewise Rational Cubic Interpolation." *Int. Journal of Math. Analysis,* vol.5, pp. 99 – 104,2011.

[10] T.N.T. Goodman. "Shape preserving interpolation by curves." *In Algorithms for Approximation IV Proceedings,* 2002, pp. 24–35.

[11] P.Lamberti and C.Manni. "Shape-preserving C^2 functional interpolation via parametric cubics." *Numer. Algorithms,* vol.28, pp. 229–254,2001.

[12] J.C.Fiorot and J.Tabka. "Shape-Preserving C^2 Cubic Polynomial Interpolating Splines." *Mathematics of Computation,* vol.57, Pages 291-298,1991.

[13] M.Shrivastava and J.Joseph. "C^2-rational cubic spline involving tension parameters." *India Proc. Indian Acad. Sci. (Math. Sci.), Vol. 110, pp. 305-314, Aug.2000.*