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Weak ss-Lifting Modules

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Abstract

Let M be a module. M is called *weak ss-lifting* if it is ss-supplemented and its ss-supplement submodules are direct summand. In this paper, we provide the basic properties of weak ss-lifting modules. In particular, we show that every direct summand of a weak ss-lifting module is weak ss-lifting. Moreover, we prove that a ring R is semiperfect with semisimple radical if and only if every projective left R-module is weak ss-lifting.

Keywords: weak (ss-) lifting module; ss-supplemented module; semiperfect ring, local commutative principle ideal ring.

1. Introduction

A submodule N of M will show that $N \subseteq M$. $Rad(M)$ and $Soc(M)$ will indicate radical and socle of M, respectively. A submodule K of M is called a *supplement* of N in M if $M = N + K$ and $N \cap K << K$. The module M is called *supplemented* if every submodule of M has a supplement in M. A submodule K of M has *ample supplements* in M if every submodule T of M such that $M = K + T$ contains a supplement of K in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M [7]. For a module M, two submodules N and K of M are called *mutual supplements* if, $M = N + K$, $N \cap K << N$ and $N \cap K << K$ [1]. In [9], Zhou and Zhang generalized the concept of socle of a module M to that of $Soc_s(M)$ by considering the class of all simple submodules of M that are small in M in place of the class of all simple submodules of M, that is,

$Soc_s(M) = \sum\{ N << M \mid N \text{ is simple}\}$. It is clear that $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$.

A module M is called *lifting*, if for any submodule N of M, there exists a direct summand K of M such that $K \subseteq N$ and $N/K << M/K$, equivalently M is lifting if and only if M is amply supplemented and every supplement submodule of M is direct summand.

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We say *N* lies above *K* in *M* if *N/K* <<*M/K*. Detailed studies about lifting modules can be found at the [1]. As a proper generalization of lifting modules, Nebiyev and Pancar defined this way strongly ⊕-supplemented modules in [5]. A supplemented module *M* is called *strongly ⊕-supplemented* if every supplement submodule of *M* is a direct summand in *M*. Then this definition is available in [1] as weak lifting modules and there exists basic properties of these modules. A submodule *K* of *M* is called a *ss-supplement* of *N* in *M* if *M* = *N* + *K* and *N* ∩ *K* ⊆ $Soc_s(K)$. The module *M* is called *ss-supplemented* if every submodule of *M* has a ss- supplement in *M*. A submodule *K* of *M* has *ample ss-supplements* in *M* if every submodule *T* of *M* such that *M* = *K* + *T* contains a ss-supplement of *K* in *M*. The module *M* is called *amply ss-supplemented* if every submodule of *M* has ample ss-supplements in *M*. The class of ss-supplemented modules was first studied by Kaynar et.al. in [4]. In this article, motivated by weak lifting modules, we define weak ss-lifting modules and provide the basic properties of weak ss-lifting modules. In particular, we show that every direct summand of a weak ss-lifting module is weak ss-lifting in Proposition 2.5. In Theorem 2.8, we characterize the rings whose projective modules are weak ss-lifting. Throughout this paper, *R* will always denote an associative ring with identity element and modules will be left unital. *Rad(R)* will denote the Jacobson radical of the ring R. We will use the notation *N*<< *M* to stress that *N* is small submodule of *M*. We refer to [1,4,7] for any undefined notion arising in the tex.

2. Results

In this section, we investigate some properties of weak ss-lifting modules. We mainly study the relation between the notion of weak ss-lifting modules and some other notions. Definition 2.1. A ss-supplemented module *M* is called *weak ss-lifting* if every ss-supplement submodule of *M* is a direct summand.

It is clear that the following implications hold:

weak ss-lifting ⟹ weak lifting

Lemma 2.2. *Let M be a weak lifting module and Rad(M)* ⊆ *Soc(M). Then M is weak ss-lifting.* Proof. *Clearly,* M *is supplemented. By [4, Lemma 19],* M *is ss-supplemented because* Rad(M) ⊆ Soc(M). *Let* V *be a ss-supplement in* M. *Therefore, we can write* M = U + V *and* U ∩V<<V *for some submodule* U ⊆ M. *Since* M *is weak lifting,* V *is a direct summand of* M. *It means that* M *is weak ss-lifting.*□ *Now we give an example which shows that a weak lifting module need not be weak ss-lifting.*

Example 2.3. Let *F* be a field, *R* = *F*[[X]] and *K* be the quotient field of the commutative domain R. Put

$M =_R K$. It is clear that *M* is a weak lifting module but *M* is not weak ss-lifting module since *Soc(M)* = 0.

It is clear that every weak ss-lifting module is ss-supplemented. Now we show that the converse is true accepting special condition in the following Lemma. Recall from [7, 41.13] that a module *M* is called *π-projective* if, for every two submodules *N*, *K* of *M* and identity homomorphism . $I_M : M \longrightarrow M$ with *M* = *N* + *K*, there exists *p* ∈ *End(M)* with *Im(p)* ⊆ *N* and $Im(I_M - p) \subseteq K$.

Lemma 2.4. Let M be a ss-supplemented and *π*-projective module. Then M is a weak ss-lifting module.

Proof. Since *M* is ss-supplemented and π-projective, *M* is amply ss-supplemented by [4, Proposition 37]. Let *K* be a ss-supplement of some submodule $N \subseteq M$. Since *M* is amply ss-supplemented, there exists a submodule N' of *N* such that $M = N + K = N' + K$ and $N' \cap K \subseteq Soc_s(N')$. It follows that N' and *K* are mutual supplements in M. By [7, 41.14(2)], we obtain that $K \cap N' = 0$. Therefore $M = N' \oplus K$. Thus *M* is weak ss-lifting.□

Proposition 2.5. Let M be a weak ss-lifting module. Then every direct summand of M is weak ss-lifting.

Proof. Given $M = U \oplus V$. By [4, Proposition 26], we deduce that *U* is ss-supplemented as a factor module of *M*. Let *K* be a ss-supplement of *N* in *U*. It is clear that *K* is a ss-supplement of $N \oplus V$ in *M*. Since *M* is weak ss-lifting, we can write $M = K \oplus W$ for some submodule $W \subseteq M$. Then Applying modular law, we get that $U = U \cap M = U \cap (K \oplus W) = K \oplus (U \cap W)$. Therefore *K* is a direct summand of *U*. Hence *U* is a weak ss-lifting module.□ *Theorem 2.6.* Let M_i be projective module for *every 1* < i < n. Then $M = \oplus_{i=1}^{n} M_i$ is weak ss-lifting if and only if every M_i is weak ss-lifting. *Proof.* (⇒) Since every M_i is a direct summand of *M* for $1 < i < n$, the proof follows from Proposition 2.5. (⇐) Since every M_i is ss-supplemented for $1 < i < n$, *M* is ss-supplemented by [4, Corollary 24]. It follows from [7] that *M* is projective. Then *M* is π-projective by [1, 4.13]. So *M* is weak ss-lifting by Lemma 2.4. *Proposition 2.7.* Let M be a π-projective module with Rad(M) ⊆ Soc(M). Then the following statements are equivalent.

1. *M is ss-supplemented;*
2. *M is weak ss-lifting;*
3. *M is supplemented;*
4. *M is weak lifting.*

Proof. (1) ⇒ (2) Since *M* is π- projective, the proof follows from Lemma 2.4.

(2)⇒(1) Clear.

(1) ⇔ (3) ⇔ (4) Clear by [4, Lemma 39].□

Recall that an R-module *M* is *semiperfect* if every factor module of *M* has a projective cover. If the ring *R* as a left R-module is semiperfect then the ring *R* is semiperfect.

Theorem 2.8. *For any ring R, the following statements are equivalent.*

1. *R is semiperfect and Rad(R) ⊆ Soc($_R$R);*
2. *$_R$R is ss-supplemented;*
3. *$_R$R is weak ss-lifting;*
4. *every projective left R-module is weak ss-lifting;*
5. *every projective left R-module is ss-supplemented.*

Proof. By Lemma 2.2 and [4, Theorem 41].□

Theorem 2.9. *For a ss-supplemented module M, the following statements are equivalent.*

(1) M is weak ss-lifting;
(2) every ss-supplement submodule of M lies above a direct summand;
(3) (a) every non-zero ss-supplement submodule of M contains a non-zero direct summand of M;

(b) every ss-supplement submodule of M contains a maximal direct summand of M.

Proof. (1) ⇒(2) Clear.

(2) ⇒(1) Let *U* be a submodule of *M* and *V* be a ss-supplement of *U* in *M*. By hypothesis, there exist submodules $K \subseteq M$ and $L \subseteq M$ such that $M = K \oplus L$, $K \subseteq V$ and $V \cap L <$ L. Then $V = V \cap M = V \cap (K \oplus L) = K \oplus (V \cap L)$, $V \cap L << M$ and $M = U + V = U + K + (V \cap L) = U + K$. Since *V* is a ss-supplement of *U*, we have $V = K$. Thus $M = V \oplus L$. Hence *M* is weak ss-lifting.

(1) ⇒ (3) Clear.

(3) ⇒ (1) Let *V* be a ss-supplement of *U* in *M*. Suppose that $W \subseteq V$ and $M = W \oplus S$. Then $V = W \oplus (V \cap S)$ and $V \cap S$ is a ss-supplement of $U + W$ in *M*. If $V \cap S = 0$, then by 3-(a) there exists a non-zero direct summand *T* of *M* such that $T \subseteq V \cap S$. Then we have $W \oplus T$ is a direct summand of *M* and $W \oplus T \subseteq V$. This contradicts the choice of *W*. Thus $V \cap S = 0$ and $V = W$. So *V* is a direct summand of *M*. Therefore *M* is weak ss-lifting. *Proposition 2.10.* Let R be a commutative noetherian local ring with maximal ideal m. The following statements are equivalent for a projective R-module M and Rad(M) ⊆ Soc(M).

(1) M is weak lifting;
(2) M is weak ss-lifting;
(3) $M \cong \oplus_{k\in K} R/J_k$ *where* J_k *are ideals of R*

(a) *there exists* $l > 1$ *such that the set* $\{ k \in K \mid m^l \not\subseteq J_k \}$ *is finite,*
(b) *the ideals* $\{ J_k \mid k \in K \}$ *are linearly ordered by inclusion, and*
(c) *if* $Ji \subseteq J_h$ *then* $mJ_h \subseteq J_i$.

Proof. Follows from Lemma 2.2 and [3, Proposition 3.7]. *Corollary 2.11.* Let R be a commutative noetherian local ring with maximal ideal m. The following statements are equivalent for a projective R-module M and Rad(M) ⊆ Soc(M).

(1) M is lifting;
(2) M is weak lifting;
(3) M is weak ss-lifting;
(4) $M = \oplus_{k\in K} Ra_k$ *and for every pair (k, n)* $\in K \times K$, $Ra_k \oplus Ra_n$ *is weak ss-lifting.* □

In the remaining indication of this article we denote $B_m(k,k+1)$ the direct sum of arbitrarily many copies of R/m^k and $R/m^{k+!}$ where m is a maximal ideal of R and k is a non-negative integer.

Theorem 2.12. *Let R be a local commutative principal ideal ring with maximal ideal m. If M is a projective R-module and Rad(M)* $\subseteq$ *Soc(M), then the following statements are equivalent:*

(1) *M is weak lifting;*
(2) *M is weak ss-lifting;*
(3) $M \cong B_m(k, k+1)$ *or* $M \cong R^{(a)}$ *for some non-negative integers a and k.*

Proof. Clear by Proposition 2.10.□

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