American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS)

ISSN (Print) 2313-4410, ISSN (Online) 2313-4402

http://asrjetsjournal.org/

On the Relationship Between the Fractional Sumudu Transform and Fractional Fourier Transform

Sadaf Saeed[a], Saleem Iqbal[b]*, Farhana Sarwar[c], Abdul Rehman[d]

[a,b,d]*Department of Mathematics, University of Balochistan, Quetta 87300, Pakistan*

[c]*Department of Mathematics F.G.Girls Degree College, Madrissa Road, Quetta, Cantt, 87300, Pakistan*

[a]*Email: sadafimran6zg@gmail.com*

[b]*Email: saleemiqbal81@yahoo.com*

[c]*Email: f_saleem10@yahoo.com*

[c]*Email: rehman_maths@hotmail.com*

Abstract

In this paper we have discussed the kernel of fractional Sumudu transforms and its relationship with fractional Fourier transform which will play a significant role for fractional Sumudu transform to recognize its importance like other fractional transforms.

Keywords: Sumudu Transform; Fractional Sumudu transform; Fractional Laplace Transform; Fractional Mellin Transform; Fractional Fourier Transform.

1. Introduction

The Sumudu transform which is introduced in 1993[1] is not as well-known as other transforms like Laplace, Fourier, Mellin, Hilbert and other transforms. Sumudu transform carries some advantages as compare to other transform and it is found capable in solving complicated problems. The fundamental properties of Sumudu transform were established in [2] and subsequently applied to partial differential equations [3-5]. The Sumudu transform which is equivalent to the Laplace transform also leads to a series of properties in a similar way like Laplace transform. The Sumudu transform may be used to solve problems without resorting to a new frequency domain [6-7].

----

* Corresponding author.

In 2003, Belgacem and his colleagues [8] have shown it to be the theoretical dual to the Laplace transform, the relationship of Sumudu transform with Laplace transform is discussed in [9]. The inverse Sumudu transform is also exist and discussed in [10]. The idea of fractional transforms attracted the researchers after the introduction of fractional Fourier transform by Namias in 1980 [11]. Now a days the mathematicians, physicists are paying their attention not only on fractional Fourier transforms but also working on many other fractional integral transforms and provided well-established and valuable methods for solving problems in many areas of applied mathematics, physics, optics, signal processing, quantum mechanics through integral transforms techniques. Fractional Sumudu transform were also introduced which is the generalization of Sumudu transform [12]. Still lot of properties of fractional Sumudu transform needs to find out, so researchers are also paying their attention to this newly introduced transform. In this paper we have established the relationship of fractional Sumudu transform with fractional Fourier transforms and find out the formula for kernel of fractional Sumudu transform.

2. Results and Discussions

In this section we are presenting the kernel of fractional Sumudu transform and its relationship with fractional Fourier transform, which can play a significant role in solving complicated problems arising in signal processing and other fields of applied mathematics and will also helpful to obtained the solutions of fractional differential equation.

2.1. The Kernel of fractional Sumudu transform (FRST)

The fractional Laplace transform are discussed in [13]. And defined as

$$\mathcal{L}^{\alpha}\,\{\phi(t)\} = \bar{\phi}^{\alpha}(s) = \int_{-\infty}^{\infty} \phi(t) k_{\alpha}(t,u)\, dt \qquad (1)$$

Where $k_{\alpha}(t,u)$ is called kernel and it is defined as

$$k_{\alpha}(t,u) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i\frac{\cot\alpha}{2}(t^2 - u^2 - 2iut\sec\alpha)} \qquad (2)$$

The classical ST is defined as

$$F(u) = \boldsymbol{S}[\varphi(t)] = \frac{1}{u}\int_{0}^{\infty} e^{-\left(\frac{t}{u}\right)}\, \varphi(t) dt \qquad (3)$$

provided the integral exists for some u. where $\boldsymbol{S}$ is the Sumudu transform operator. Since there is a close relation between Laplace and Sumudu transform the relationship can be written as

$$F\left(\frac{1}{s}\right) = S[\varphi(t)] = s\int_{0}^{\infty} e^{-st}\, \varphi(t) dt = sG(s) \qquad (4)$$

Where $G(s)$ is the Laplace transform of $\varphi(t)$

The fractional Sumudu transform discussed and defined in the following manner in [12]

$$F_\alpha\left(\frac{1}{s}\right) = s^\alpha G_\alpha(s), \qquad 0 < \alpha < 1 \qquad (5)$$

or

$$F_\alpha(u) = \frac{1}{u^\alpha} G_\alpha\left(\frac{1}{u}\right), \quad 0 < \alpha < 1 \qquad (6)$$

where G_α is the fractional Laplace transform and defined as

$$G_\alpha(u) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} \varphi(t) e^{\frac{i\cot\alpha}{2}(t^2 - u^2) - tu\csc\alpha} dt \qquad (7)$$

From equations (6) and (7) the integral formula of fractional Sumudu transform is defined as

$$F_\alpha(u) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}} \frac{1}{u^\alpha} \int_{-\infty}^{\infty} \varphi(t) e^{\frac{i\cot\alpha}{2}\left(t^2 - (\frac{1}{u})^2\right) - \frac{t}{u}\csc\alpha} dt \qquad (8)$$

$$k_\alpha\left(t, \frac{1}{u}\right) = \frac{1}{u^\alpha} \sqrt{\frac{1 - i\cot\alpha}{2\pi}} e^{\frac{i\cot\alpha}{2}\left(t^2 - (\frac{1}{u})^2\right) - \frac{t}{u}\csc\alpha} \qquad (9)$$

The kernel of fractional Sumudu transform is defined in equation (8) Since there is a close relation between fractional Laplace and fractional Sumudu transform [13] and the relationship between FRST with FRFT discussed in this article the kernel properties will be followed as discussed in [14].

2.2. The relationship between fractional Sumudu transform with fractional Fourier Transform

Since fractional Fourier transform of $\phi(t)$ is defined as [14]

$$\mathcal{F}^\alpha\{\phi(t)\} = \overline{\varphi(t)}^\alpha(\omega)$$

$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} \phi(t) e^{\frac{i\cot\alpha}{2}(t^2 + \omega^2) - (i\omega)t\csc\alpha} dt, \text{ when } \alpha \text{ is not a multiple of } \pi \qquad (10)$$

Replace u by $\frac{1}{s}$ in eqn (8) we get,

$$F^\alpha\left(\frac{1}{s}\right) = s\,\bar{\phi}^\alpha(s) = s^\alpha \sqrt{\frac{1 - i\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} \varphi(t) e^{\frac{i\cot\alpha}{2}(t^2 - s^2) - ts\csc\alpha} dt$$

since s is complex therefore substituting $s = \sigma + i\omega$ in equation (6) we get

$$G^{\alpha}\left(\frac{1}{\sigma+i\omega}\right)=(\sigma+i\omega)^{\alpha}\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\varphi(t)e^{\frac{i\cot\alpha}{2}(t^2-(\sigma+i\omega)^2)-(\sigma+i\omega)t\csc\alpha}dt$$

$$=\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\varphi(t)e^{\frac{i\cot\alpha}{2}(t^2-\sigma^2+\omega^2-2i\sigma\omega)-(\sigma+i\omega)t\csc\alpha}dt$$

$$=e^{-\frac{i\cot\alpha}{2}(\sigma^2+2i\sigma\omega)}\sqrt{\frac{1-i\cot\alpha}{2\pi}}(\sigma+i\omega)\int_{-\infty}^{\infty}\varphi(t)e^{-\sigma t\csc\alpha}e^{\frac{i\cot\alpha}{2}(t^2+\omega^2)-(i\omega)t\csc\alpha}dt$$

$$=e^{-\frac{i\cot\alpha}{2}(\sigma^2+2i\sigma\omega)-\sigma t\csc\alpha}(\sigma+i\omega)^{\alpha}\left[\sqrt{\frac{1-i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}(\phi(t)e^{-\sigma t\csc\alpha})e^{\frac{i\cot\alpha}{2}(t^2+\omega^2)-(i\omega)t\csc\alpha}dt\right] \quad (11)$$

using (8) and (10) we get the result from eqn (10)

$$\boldsymbol{S}^{\alpha}\{\varphi(t)\}=e^{-\frac{i\cot\alpha}{2}(\sigma^2+2i\sigma\omega)-\sigma t\csc\alpha}(\sigma+i\omega)^{\alpha}\mathcal{F}^{\alpha}\{\varphi(t)e^{-\sigma t\csc\alpha}\} \quad (12)$$

Eq (12) shows the relationship of fractional Sumudu transform with fractional Fourier transform

3. Conclusion

We have established relationship between fractional Sumudu transform and fractional Fourier transform and find the of Kernel of fractional Sumudu which will play a significant role to recognized the importance of fractional Sumudu transform as compare to fractional Fourier transform.

Reference

[1]. G. K. Watugala, "Sumudu transform: a new integral transform to solve differential equations and control engineering problems" Int. J. Math. Educ. Sci. Technol., Vol. 24, No. 1, 35-43 , 1993

[2]. F.B.M Belgacem, A.A Karaballi "Sumudu transform fundamental properties investigations and applications. International Journal of Stochastic Analysis. Vol. 2006; 1-23, 2006.

[3]. M. A. Asiru, "Further properties of the Sumudu transform and its applications," International Journal of Mathematical Education in Science and Technology, vol. 33, no. 3, pp. 441–449, 2002.

[4]. A. Kılıc¸man andH. Eltayeb, "A note on integral transforms and partial differential equations," Applied Mathematical Sciences, vol. 4, no. 1–4, pp. 109–118, 2010.

[5]. A. K,¸man, H. Eltayeb, and P. R. Agarwal, "On Sumudu transform and system of differential equations," Abstract and Applied Analysis, vol. 2010, 2010.

[6]. Keji Li, Y. X. (). A brief introduction of Sumudu transform and comparison with other integral

transforms. IEEE, 285-287. 2011,

[7]. V. G. Gupta and B. Sharma, "Application of Sumudu transform in reaction-diffusion systems and nonlinear waves," Applied Mathematical Sciences, vol. 4, no. 9–12, pp. 435–446, 2010.

[8]. F. B. M. Belgacem, A. A. Karaballi, and S. L. Kalla, "Analytical investigations of the Sumudu transform and applications to integral production equations," Mathematical Problems in Engineering, no. 3-4, pp. 103–118, 2003 .

[9]. M. Baloch., S. Iqbal, The Relationship of Fractional Laplace Transform with Fractional Fourier, Mellin and Sumudu Transforms. American Scientific Research Journal for Engineering, Technology, and Sciences (ASRJETS), 55(1), pp.11-16., 2019

[10]. R. Silambarasan, K.S Nisar, et al "Discrete inverse Sumudu transform application to Whittaker and Zettl equations. AIP Conference Proceedings ,Vol. 2046, No. 1, p. 020091, 22018

[11]. V. Namias, "The fractional order Fourier transform and its application to quantum mechanics", IMA Journal of Applied Mathematics, 25(3), 241-265, 1980.

[12]. Gupta VG, Shrama B, Kiliçman A. A note on fractional Sumudu transform. Journal of Applied Mathematics, 2010

[13]. K.K Sharma. Fractional Laplace transform, Springer-Verlag London, 377-379, 2010

[14]. L.B. Almeida, "The fractional Fourier transform and time-frequency representations", IEEE Transactions on Signal Processing, 42(11) , 3084–3091, 1994