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ℚ is a Convergence Set

Basma Al-Shutnawi[a*], Mohammad Zannon[b]

[a] Department of Mathematics, Tafila technical university, P.O. Box 179 , Tafila 66110, Jordan.

[b] Department of Mathematics, Tafila technical university, P.O. Box179, Tafila 66110, Jordan.

[a] basma@ttu.edu.jo

[b]zanno1ms@gmail.com

Abstract

In this paper we consider the convergence sets of formal power series of the form $f(z,t) = \sum_{j=0}^{\infty} f_j(z)t^j$, where $f_j(z)$ are polynomials functions on a domain Ω in $\mathbb{C}$. A subset E of Ω is said to be convergence set if there is a series $f(z,t)$ such that E is exactly the set of points z for which $f(z,t)$ converges as a power series in t in some neighborhood of the origin. We prove that $\mathbb{Q}$ is a convergence set.

Keywords: formal power series; convergence sets, $\mathbb{Q}$ the set of rational numbers, quasi-simply-connected sets.

1. Introduction

Let $\mathbb{C}$[z] be the set of polynomials in z, $\mathbb{C}[[z]]$ be the set of formal power series, and $\mathbb{C}[z][[t]]$ be the set of formal power series in t with coefficients being polynomials in z. In my dissertation [3] I considered the formal power series

$$\boldsymbol{f(z) = a_0 + \sum_{|\alpha_1|=1} a_{\alpha_1} z^{\alpha_1} + \sum_{|\alpha_2|=2} a_{\alpha_2} z^{\alpha_2} + \cdots + \sum_{|\alpha_n|=n} a_{\alpha_n} z^{\alpha_n},}$$

Where $\alpha_j = (\alpha_{j1},\ldots, \alpha_{jn})$ are the n-multiple index and $|\alpha_j| = |\alpha_{j1}| + \cdots + |\alpha_{jn}|$. Many research concerning convergence (or formal) series [7, 1, 11, 8, 5]. The general description of these problems is given in [12]. Recently there were new researches concerning the power series when the coefficients are polynomials of two or more complex variables [2, 4, 9, 6]. Consider $F(z,t) = \sum_{m=0}^{\infty} P_m(z)t^m$. Suppose that $F(z,t)$ as a power series in t converges for z in a set $E \subset \mathbb{C}^n$. I mentioned in my dissertation that the set $\mathbb{Q}$ of rational numbers is a convergence set in this paper we prove this corollary.

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* Corresponding author.

E-mail address: basma@ttu.edu.jo.

2. $\mathbb{Q}$ is a Convergence Sets

2.1. Definition [3]

A power series $f \in \mathbb{C}[[z_1, \ldots, z_n]]$ is said to be convergent if there is a constant *C* such that $|a_{k1,\ldots,kn}| \leq C^{k1+\cdots+kn}$ for all $(k_1, \ldots, k_n) \neq (0, \ldots, 0)$.

A power series f is said to be divergent if it is not convergent.

2.2. Definition [3]

Let $f(z,t) \in \mathbb{C}[z][[t]]$. Define the convergence set of f to be *Conv(f) = {z ∈ ℂ : f(z,t) converges in t}.*

2.3. Definition [3]

A subset $E \in \mathbb{C}$ is said to be a convergence set if there exists an $f \in \mathbb{C}[z][[t]]$ such that E = *Conv(f).*

Moreover we proved the following theorem,

2.4. Theorem [3] Let $S = \{z_1, z_2, \ldots\}$ be a countable infinite subset of $\mathbb{C}$. Define an F ∈ ℂ[z][[t]] by

$$\boldsymbol{F(z,t) = \sum_{n=0}^{\infty} C_n [\prod_{j=1}^{n} (z - z_j)]\ t^n},$$

Where $C_n = ({}^{n}/_{\gamma_n})^n$, and

$$\boldsymbol{\gamma_n = \min\left(\frac{1}{2} \min_{1 \leq j \leq n+1} |z_i - z_j|,\ 1/n\right)}.$$

Then *Conv(F)* = S.

As a corollary from this theorem we mentioned that the set of rational numbers is a convergence set, we discuss the proof in this paper.

2.5. Corollary

The set $\mathbb{Q}$ of rational numbers is a convergence set.

Proof. $For\ n \in \mathbb{N}$, by Weierstrass theorem see [10] any function on the closed disc $\{z \in \mathbb{C} : |z| \leq n\}$ can be approximated arbitrarily by a polynomial. So for $(n^n \sin n! \pi z)$, $n \in \mathbb{N}$, one can find a polynomial P_n(z) such that

$$\boldsymbol{|P_n(z) - n^n \sin n! \pi z| < \frac{1}{n}, \text{ for } |z| \leq n}.$$

Let the formal series

$\boldsymbol{F(z,t) = \sum_{n=0}^{\infty} P_n(z)t^n}$.

Now for every $z \in \mathbb{C}/\mathbb{Q}$, the bounded sequence $\{\sin n!\pi z)\}_{n=1}^{\infty}$ is divergent as n extends to infinity. Suppose there exist a sub-sequence of $\{n_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} \sin n!\pi z = \lambda$, where λ is a complex number. Using the previous inequality we get

$|\boldsymbol{P_{nj}(z) - n_j^{nj} \sin n_j! \pi z)}| < \frac{1}{n_j}$, **for j ≤** $|z|$,

Which gives that

$|\boldsymbol{P_{nj}(z)}| \geq |\boldsymbol{n_j^{nj} \sin n_j! \pi z}| - \frac{1}{n_j}, \forall$ **j ≥** $|z|$.

$|\boldsymbol{P_{nj}(z)} \geq \frac{1}{2}|\boldsymbol{\lambda}|\boldsymbol{n_j^{nj}}$,

For n_j large enough, $F(z,t)$ is divergent $\forall n \in \mathbb{N}$.

On the other hand, for x∈ $\mathbb{Q}$ let x = $\frac{l}{m}$ where $l, m \in \mathbb{Z}$ and the $\gcd(l,m) = 1$. Then for $n > m$

$|\boldsymbol{P_n(x)}| \leq |\ \boldsymbol{n^n \sin n! \pi x}) + \frac{1}{n}| = \boldsymbol{0} + \frac{1}{n}$.

So $F(x,t)$ is convergence.

The Corollary indicates that the whole rational numbers $\mathbb{Q}$ is a convergence set. Now

Since the unit interval [0, 1] is compact and simply connected set it's a convergence set. In [3] we proved that a finite intersection of convergence sets is a convergence set, so the intersection of the $\mathbb{Q}$ and the unit interval is a convergence set.

2.6. Example

The set of rational number in the unit interval, K = [0, 1] \ $\mathbb{Q}$, is a convergence set.

3. Conclusion

In this paper we find that $\mathbb{Q}$ is a convergence set, which implies that $\mathbb{Q}$ is a quasi-simply-connected set, looking for new sets which is convergence and have other formal power series properties.

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