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Revisiting the Hodrick-Prescott Filter: A New Perception of the Minimization Problem

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Abstract

When we revisit how reflect the objective of reducing the effects of the growth rates fluctuation in the Hodrick-Prescott filter, we minimize the gap between the value of the trend at time t and a third order average around that value. The paper assesses that the obtained modified Hodrick-Prescott filter better takes into account the structure of the series and gives a stronger estimate of the turning points.

Keywords: Economic cycle; Hodrick-Prescott Filter; Smoothing parameter; Turning points.

1. Introduction

Macroeconomic theory of cycle seeks to explain the individual and common cyclical movement for economic variables in order to provide economic analysis and policy making. To analyze cycles in observed data it is necessary to isolate the cyclical component from the trend. This decomposition is widely achieved by regression or filtering method such as the Baxter and King filter ([2]), the Christiano and Fitzgerald ([3]) and the Hodrick and Prescott filter ([7]). The latter, the so-called HP filter, was criticized for the border distortion and the modification of the turning points chronology ([4, [10]). Nonetheless, the HP filter remains the most used in applied macroeconomic work by economists in central banks, international economic agencies, industry, and government and serves as a benchmark for other methods of cycle extraction. In addition, a renewed interest is emerging in the HP filter. The authors in [8] showed that the HP filter extracts a cyclical component of periodicity ranging from 1 to10 years, without modifying the smoothing period. The authors in [10] provide exact matrix and operator representations and new asymptotics. The author in [5] gives an exact analytical expression for the weights in the HP filter. The authors in [6,11] provide further finite sample results, including another representation of the HP filter as a symmetric weighted average plus some adjustments.

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The present paper presents the first-order conditions of the HP Filter minimization problem and highlights a new perception of this minimization problem. A new derivation of the HP minimization problem's second part that is developed, which leads to a modified filter hereafter called HPmod. It is an easy-to-implement modification that is designed to make the HP filter more effective in extracting cycle with better identified turning points chronology.

2. The modified Hodrick-Prescott filter

2.1. The Hodrick-Prescott filter

The HP filter decomposes an observed series Y_t, into X_t a non-stationary long-term trend and C_t a stationary short-term cyclical residual, with time $t = 1, 2, \dots n$. X_t is the solution to the following minimization problem :

$$\underset{(X_t)}{Min} \sum_{t=1}^{n}(Y_t - X_t)^2 + \lambda \sum_{t=2}^{n-1}(X_{t-1} - 2X_t + X_{t+1})^2 = \underset{(X_t)}{Min} \sum_{t=1}^{n}[(Y_t - X_t)^2 + \lambda(\Delta^2 X_t)^2] \tag{1}$$

where λ the smoothing parameter. The HP first-order conditions are derived by setting the gradient vector of equation (1) equals to zero. We obtained :

$C_1 = \lambda(x_1 - 2x_2 + x_3)$
$C_2 = \lambda(-2x_1 + 5x_2 - 4x_3 + x_4)$
$C_t = \lambda(x_{t-2} - 4x_{t-1} + 6x_t - 4x_{t+1} + x_{t+2})\ , \quad for\ t = 3,4,5, \dots, N-2$
$C_{N-1} = \lambda(x_{N-3} - 4x_{N-2} + 5x_{N-1} - 2x_N)$
$C_N = \lambda(x_{N-2} - 2x_{N-1} + x_N)$

Or more compactly, $C = \lambda FX$.

$$F = \begin{bmatrix} 1 & -2 & 1 & 0 & \cdots & \cdots & \cdots & & \cdots & 0 \\ -2 & 5 & -4 & 1 & 0 & & & & & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & & & & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & & & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \cdots & 0 \\ \vdots & & & & & & & & & \vdots \\ \vdots & & & & & & & & & \vdots \\ 0 & \cdots & 0 & 1 & -4 & 6 & -4 & 1 & 0 & 0 \\ 0 & & & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & & & & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & & & & & 0 & 1 & -4 & 5 & -2 \\ 0 & \cdots & & & \cdots & \cdots & 0 & 1 & -2 & 1 \end{bmatrix}$$

Informations mentionned above implies that $Y = (\lambda F + I)\hat{X}$. The HP trend is $\hat{X}_t = M^{-1}Y_t = (\lambda F + I)^{-1}Y_t$ and $\hat{C} = Y - \hat{X}$, except for the four endpoints, the first-order conditions state that $C_t = \lambda\Delta^2 X_t$, for $t = 3, \dots, N-2$.

2.2. A new alternative derivation of the HP minimization problem

In the HP filter, we minimize the gap between the value of the trend at time t and the first order moving average centered at time t. $\Delta^2 X_t = X_{t-1} - 2X_t + X_{t+1} = 3\left(\frac{X_{t-1}+X_t+X_{t+1}}{3} - X_t\right)$

Thus, we suggest to reach the objective of reducing the effects of the growth rates fluctuation reflected by the trend when minimizing the gap between the trend value at time t and the second order average around that value. The HPmod filter uses a longer moving average, which is minimized to prevent large oscillation of X_t.
We named this moving average : $(D_2M)X_t$.
$(D_2M)X_t = (X_{t+2} - X_t) - (X_t - X_{t-2}) + (X_{t+1} - X_t) - (X_t - X_{t-1}) = \sum_{j=-2}^{2} X_{t+j} - 5X_t$.

The minimization problem is:

$$\underset{(X_t)}{Min} \sum_{t=1}^{n} [(Y_t - X_t)^2 + \lambda((D_2M)X_t)^2] \qquad (2)$$

The first-order conditions are:

$C_1 = \lambda(x_1 + x_2 - 4x_3 + x_4 + x_5)$
$C_2 = \lambda(x_1 + 2x_2 - 3x_3 - 3x_4 + 2x_5 + x_6)$
$C_3 = \lambda(-4x_1 - 3x_2 + 18x_3 - 7x_4 - 7x_5 + 2x_6 + x_7)$
$C_4 = \lambda(x_1 - 3x_2 - 7x_3 + 19x_4 - 6x_5 - 7x_6 + 2x_7 + x_8)$
$C_t = \lambda(x_{t-4} + 2x_{t-3} - 7x_{t-2} - 6x_{t-1} + 20x_t - 6x_{t+1} - 7x_{t+2} + 2x_{t+3} + x_{t+4})\ , \quad for\ t = 5, 6, \dots, N-4$
$C_{N-3} = \lambda(x_{N-7} + 2x_{N-6} - 7x_{N-5} - 6x_{N-4} + 19x_{N-3} - 7x_{N-2} - 3x_{N-1} + x_N)$
$C_{N-2} = \lambda(x_{N-6} + 2x_{N-5} - 7x_{N-4} - 7x_{N-3} + 18x_{N-2} - 3x_{N-1} - 4x_N)$
$C_{N-1} = \lambda(x_{N-5} + 2x_{N-4} - 3x_{N-3} - 3x_{N-2} + 2x_{N-1} + x_N)$
$C_N = \lambda(x_{N-4} + x_{N-3} - 4x_{N-2} + x_{N-1} + x_N)$

Or more compactly $Y = (\lambda H + I)\hat{X}$.

$$H = \begin{bmatrix}
1 & 1 & -4 & 1 & 1 & 0 & \cdots & \cdots & \cdots & & & & & & 0 \\
1 & 2 & -3 & -3 & 2 & 1 & 0 & & & & & & & & 0 \\
-4 & -3 & 18 & -7 & -7 & 2 & 1 & 0 & & & & & & & 0 \\
1 & -3 & -7 & 19 & -6 & -7 & 2 & 1 & 0 & & & & & & 0 \\
1 & 2 & -7 & -6 & 20 & -6 & -7 & 2 & 1 & 0 & & & & & 0 \\
0 & 1 & 2 & -7 & -6 & 20 & -6 & -7 & 2 & 1 & 0 & & & & 0 \\
 & \ddots & & & & & & & & & & & \ddots & & \\
\vdots & & & & & & & & & & & & & & \vdots \\
\vdots & & & & & & & & & & & & & & \vdots \\
 & & & \ddots & & & & & & & & & & \ddots & \\
0 & & & & 0 & 1 & 2 & -7 & -6 & 20 & -6 & -7 & 2 & 1 & 0 \\
0 & & & & & 0 & 1 & 2 & -7 & -6 & 20 & -6 & -7 & 2 & 1 \\
0 & & & & & & 0 & 1 & -3 & -7 & 19 & -6 & -7 & 2 & 1 \\
0 & & & & & & & 0 & -4 & -3 & 18 & -7 & -7 & 2 & 1 \\
0 & & & & & & & & 0 & 1 & 2 & -3 & -3 & 2 & 1 \\
0 & & & & & & \cdots & \cdots & \cdots & 0 & 1 & 1 & -4 & 1 & 1
\end{bmatrix}$$

3. The HPmod smoothing parameter

In order to choose the smoothing parameter for the HPmod filter, we use a criteria of equivalence between the

HPmod filter and the HP filter. It is important to bear in mind that equivalence does not imply the same spectral properties, but only the same cut-off frequency. The cut-off frequency is defined as the frequency at which the HP filter gain function has the only inflection point.

$$G_X^0(\omega) = \frac{1}{1+4\lambda(1-cos\omega)^2} \text{ and } \frac{\partial^2 G_X^0}{\partial^2\omega}(\omega_0) = 0 => G_X^0(\omega_0) = 0.5 \quad (3)$$

Based on equation (3), a “modified” cut-off frequency, that is defined as the point where $G_X^0(\omega_0) = 0.5$, will be used throughout the paper as an equivalence parameter.

Equation (2) is equivalent to:

$$\underset{(X)}{Min} \sum_{t=1}^{n}(Y_t - X_t)^2 \quad + \lambda \; \sum_{t=1}^{n}((D_2M)X_t)^2 \text{ , où } (D_2M)X_t = M(L)X_t = (1 + L - 4L^2 + L^3 + L^4)X_t \quad (4)$$

where L (or F) is the lag (or forward) operator. The term D_2M is expressed as a moving average process.

As cyclical and trend components are unobserved, we assume that :

1. $C_t = (Y_t - X_t) \sim N(0, \sigma_C^2)$
2. $x_t = (D_2M)X_t \sim N(0, \sigma_x^2)$
3. C_t and $(D_2M)X_t$ are independents, and
4. $\lambda = \frac{\sigma_C^2}{\sigma_x^2}$, σ_C^2 and σ_x^2 are known.

The autocovariance generating function (ACGF) of X_t can be written : $g_X(L) = \sigma_x^2[M(L)M(F)]^{-1}$

Thus, the estimate of X_t in equation (4) is : $\widehat{X_t} = \frac{g_X(L)}{g_Y(L)} Y_t = \frac{g_X(L)}{g_X(L) + \sigma_C^2} Y_t = \frac{1}{1+g_X(L)^{-1}. \sigma_C^2} Y_t$

$$\widehat{X_t} = \frac{1}{1+\lambda[M(L)M(F)]} Y_t \quad (5)$$

We can easily verify that $(D_2M)^2 = M(L)M(F)$. Using the Fourier transform in equation (5), it can be seen that the gain of the trend filter is given by : $G_X(\omega) = \frac{1}{1+\lambda[20-12\cos(\omega)-14\cos(2\omega)+4\cos(3\omega)+2\cos(4\omega)]}$

Now, we can determine the equivalent HPmod filter. We are looking for a parameter that respect our equivalence criteria (equation (3)) and such as $G_X(\omega_0) = G_X^0(\omega_0)$. The only solution is $\lambda = 64.645$.

4. Application of the HPmod filter

4.1. Application on a simulated series

Let the following series be considered : $Y_t = C_t + \varepsilon_t$

with $C_t = sin\left(\frac{2\pi t}{24}\right) - 0.15\, sin\left(\frac{2\pi t}{6}\right)$, for $t = 1, \dots, 120$. (6)

In the first simulations (Table 1), we consider an irregular component $\varepsilon_t \sim BB(0, \sigma_\varepsilon^2)$ and C_t is a cyclical component with two periods 1.5 years and 6 years. Observations are supposed quarterly and so described on 30 years. The cyclical standard error is 0.718, thus we choose to simulate 1000 series ε_t for standard errors smaller than this value : $\sigma_\varepsilon^2 = 0.2$. In the second simulations (Table 2), we consider an irregular component $\varepsilon_t \sim (1 - k)\emptyset(\sigma^2) + k\frac{\emptyset(\sigma^2)}{h}$ and C_t is defined in equation (6). We propose indexes of quality in order to evaluate filters performance to reproduce the correct cycle. They are average value for whole simulations. The smaller index (in absolute value) is, the better it is.

- Index of peaks (I(P)) and Index of troughs (I(Tr)). They identify cycle's turning points, we consider a point on the horizontal axis at 0, as indicating there is no deviation. We call peak (respectively trough), the larger deviation among those above (respectively below) the zero-axis before graph passes below (above) the zero axis. Given a series x_t, let $t_{1,i}^{(x)}$ be the quarter where the ith peak appears and let $t_{2,i}^{(x)}$ be the quarter where the i^{th} trough appears. *C* denotes the original cycle (simulated cycle) and $\hat{C}$ denotes the estimated cycle. There are *a* peaks and *b* troughs.

$I(P) = \frac{\sum_{i=1}^{a}\left| t_{1,i}^{(\hat{C})} - t_{1,i}^{(C)} \right|}{a}$ and $I(Tr) = \frac{\sum_{i=1}^{b}\left| t_{2,i}^{(\hat{C})} - t_{2,i}^{(C)} \right|}{b}$. A large index implies a large phase shift.

- Index of recessions (I(R)) and index of expansions (I(E)). (I(R) (respectively I(E)) is defined as the average gap between recessions duration (respectively expansions duration) of the original cycle (simulated cycle) and the estimated cycle. We can easily verify that the duration of one expansion's phase is :

$$\left(t_{1,i}^{(\hat{C})} - t_{1,i}^{(C)}\right) - \left(t_{2,i}^{(\hat{C})} - t_{2,i}^{(C)}\right).$$

- Index of global adjustment. It equals to $1 - corr\left(C_t - \hat{C}_t\right)$.

Tables (1) and (2) summarize indexes of quality for HP and HPmod filters.

Table 1: Index of quality ($\sigma_\varepsilon^2 = 0.2$)

	HP	HPmod
I(P)	0.60	0.54
I(T)	0.44	0.44
I(R)	-1.36	-1.24
I(E)	1.15	1.15
Index of global adjustment	0,03944	0,03803

Table 2: Index of quality ($\sigma_\varepsilon^2 = 0.4$)

	HP	HPmod
I(P)	0,55	0,50
I(T)	0.10	0.12
I(R)	-1,19	-0,88
I(E)	0.63	0.44
Index of global adjustment	0,03093	0,02803

In both tables, I(P) are different from 0 and I(E) are positive. The chronology of peaks is often in advance. I(Tr) are different from 0 and I(R) are negative. The chronology of troughs is often lagged. The chronology of peaks is much better with the HPmod filter, in both examples. Index of global adjustment is low for both filters and is lightly better for HPmod filter. We can conclude that the HPmod filter provides a better estimation of the turning points dates and of the cyclical component.

4.2. Application on US GDP quarterly data

In this section, we extract cycle from seasonally adjusted quarterly U.S. real GDP from 1947:Q1 to 2013:Q2, as did in [5]. Since we have to extract cycle from true data, we have to deal with the end-point bias which is a key concern of policy makers. Therefore it is clearly important to determine if the HPmod filter outperforms or not the HP filter un estimating turning points dates. As discussed in [2, [3, [13, the end-point bias is not a characteristic of the HP filter only. Matrix F and H are symmetric respectively for $t = 3, n - 2$ and for $t = 5, n - 4$. Then, the HP first-order conditions reveal an end-points bias in cyclical component's estimation for $t = 1, 2, n - 1, n$. And, for the HPmod filter, we notice an end-points bias in cyclical component's estimation for $t = 1, 2, 3, 4, n - 3, n - 2, n - 1, n$. The author in [5] recommended to assign a greater smoothing parameter to end-weights in order to solve the end-point bias, since a large smoothing parameter reduces the weights. In Tables 3 and 4, we compare peak dates estimated by the HPmod filter to those from NBER and HP filter obtained in [5]. At the end-points, the author in [5] assigned a smoothing parameter equals to 150000, which is equivalent to 6060.46875 for HPmod filter. We consider that a peak is the point where two consecutive increases in cycle are followed by a decline and a trough is the point where two consecutive declines in cycle are followed by an increase ([14, [5]). For each filter, we mention our proposed Index of peaks and Index of troughs.

Table 3: NBER peak dates and estimated peak dates

NBER	HP	HPmod
1953:Q2	1953:Q1	1953:Q1
1957:Q3	1955:Q3	1957:Q1
1960:Q2	1960:Q1	1959:Q3
1969:Q4	1968:Q2	1969:Q1
1973:Q4	1973:Q2	1973:Q3
1980:Q1	1978:Q4	1979:Q1
1981:Q3	1981:Q1	1981:Q2
1990:Q3	1989:Q2	1990:Q1
2001:Q1	2000:Q2	2000:Q3
2007:Q4	2007:Q3	2007:Q4
I(P)	3.4	1.9

Table 4: NBER trough dates and estimated trough dates

NBER	HP	HPmod
1954:Q2	1954:Q2	1954:Q2
1958:Q2	1958:Q1	1958:Q2
1961:Q1	1960:Q4	1961:Q1
1970:Q4	1970:Q4	1971:Q2
1975:Q1	1975:Q1	1975:Q2
1980:Q3	1980:Q3	1980:Q1
1982:Q4	1982:Q4	1982:Q4
1991:Q1	1991:Q1	1991:Q3
2001:Q4	2001:Q4	2001:Q4
2009:Q2	2009:Q2	2009:Q2
I(T)	0.2	0.7

The HPmod filter better estimates peak dates than the HP filter. We obtain the opposite result for estimated trough dates. Both filters provide better estimated trough dates. In general, the HPmod filter provides estimated turning points dates which are identical or much closer to the NBER turning points dates, with fewer revisions for recent periods. Hence, we can conclude that the HPmod filter is reliable in producing a more stable cycle. Assigning a different smoothing parameter at the end-points appears as an efficient way to reduce end-points bias.

5. Conclusion

In the HP filter minimization problem, the second term measures the smoothness for the growth rate of the trend component. Making similar interpretations to it, we use the difference between trend at time t and its first order moving average centred at time t. The first-order conditions reveal an end-points bias in cyclical component's estimation for the three first and last points, which requires to take into account a method to correct the end-point bias while we apply our filter. And We illustrate the performance of our filter on its ability to estimate the true cycle for two simulated series and on its ability to estimate the turning point dates for NBER peak and trough dates. The HPmod filter provides better result compared to the usual HP filter.

6. Recommendations

The HPmod filter uses a longer moving average. This property ensures good cycle's estimation and good estimation of turning points dates. Therefore, the first-order conditions reveal an end-points bias in cycle's estimation for the four first and four last points (instead of two first and two last points for the HP filter), which requires to take greater into account a data dependent method to treat the end-point bias while we apply our filter.

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