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Differentiation Property of Fractional Hankel Transform of a Function Involving Higher Order Derivatives

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Abstract

In engineering mathematics, integral transform is a widely used tool for solving linear differential equations, In recent times the newly born fractional Hankel transform has been started for playing a very important role in various fields of applied mathematics and physics like fractional Fourier transform. This paper represent a formalization of differentiation property of a function invoving high order derivatives of newly introduced fractional Hankel transform. The differentiation property is proved for different higher differential equations.

Keywords: Hankel Transform; Fractional Hankel Transform; Higher order Derivatives; Besssel's Function.

1. Introduction

The fractional Fourier transform(FrFT) is undoubtedly one of the most valuable and powerful tools now a days in optics, signal communications and applied mathematics. The fractional Fourier transforms was properly introduced by Namias in 1980 and he established the mathematical formulation and identify the eigenvalues eigenfunctions and find its applications in Quantum mechanics and used the differential property to solve the Schrodinger differential equations [1].

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After the introduction of FrFT which the generalized form of classical Fourier transform, McBride and Kerr contributed interms of mathematical formulation [2] and Almaida applied the FrFT in time frequency plane [3]. Soon after the introduction of fractional order Fourier transform, it is then become a natural thought for the scientists and mathematicians whether other transforms could also be fractionalized or not. Keeping such thoughts the fractional Hankel transform was introduced by Namias and he derive the fractional integral transforms corresponding to the Hankel transform [4]. However, in case of fractional Hankel transform the range of the order is $-1 < a < 1$ which is unlike to FrFT [1]. The Hankel transformation arises in connection with the radial part of the Laplacian operator expressed in cylindrical polar co-ordinates. The operational calculus based on the conventional Hankel transform is somewhat limited in that most of its practical applications stem from the operational relation. Fractionalization of the Hankel transforms gives rise to several useful operational relations and applications [4-9].

1.1 The Fractional Hankel transform (FRHT)

The Hankel transform and its inverse is defined by the following pair of Bessel order v [5]

$$g(k) = \int_0^\infty f(r) J_v\,(kr) r dr \qquad (1)$$

$$f(r) = \int_0^\infty g(k) J_v(kr)\, kdk \qquad (2)$$

Since eqns (1) and (2) are self -reciprocal., therefore the above pair of Hankel transforms can be in a single operator form

$$\mathcal{H}_\pi f(r) = \mathcal{H}_{-\pi} f(r) == \int_0^\infty f(r') J_v(rr')\, r' dr' \qquad (3)$$

The classical Hankel transform pair is clearly correspond to angle say $\alpha = \pi$ or $\alpha = -\pi.$ In order to define the fractional Hankel transform Namias [4] consider the fractional order of the fractional Hankel as

$$n = \frac{\alpha}{\pi} \qquad (4)$$

The classical Hankel transform have the order 1 or -1 and are self-reciprocal. The fractional Hankel transforms are not self-reciprocal and $\mathcal{H}_{-\alpha}$ is established as the inverse transform of $\mathcal{H}_\alpha$. The range of real fractional order is $-1 \le n \le +1$ unlike fractional Fourier transform whose range covered by $-2 \le n \le +2$. Namias in 1980[4] and Kerr in 1991[5] worked on the integral representation of the fractional Hankel transform and its inverse transform the pair is give below

$$H_\alpha[\,f(x)](y) = \int_0^\infty f(x)\,K_\alpha(x,y)dx \qquad (5)$$

The kernel of the transform is defined as

$$K_\alpha(x,y) = \begin{cases} A_{v,\alpha}\exp\left[-\frac{i}{2}\,(x^2+y^2)cot\frac{\alpha}{2}\right]\left[\frac{xy}{|sin\frac{\alpha}{2}|}\right]^{\frac{1}{2}} J_v\left[\frac{xy}{|sin\frac{\alpha}{2}|}\right] \\ \delta(x-y) \qquad \text{for } \alpha = 0 \;\; \alpha = 2\pi \end{cases} \qquad (6)$$

and $A_{v,\alpha} = |sin\frac{\alpha}{2}|^{-\frac{1}{2}}\exp\left[i\left(\frac{\pi}{2}\,\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)\right] \qquad ,\hat{\alpha} = sin\alpha, \alpha\epsilon R \;\&\; v > -1 \qquad (7)$

2. Results and Discussions

In this section we are presenting the differential properties of higher derivative functions which will be very important to solve the high order differential equations by using the fractional Hankel transform.

2.1 Differentiation property of fractional Hankel transform of f involing first and 2nd order derivatives

By inserting $\frac{df}{dx}$ instead of $f(x)$ in eqn (5) which is the integral representation of fractional Hankel transform eqn one can easily derive operational relation transform $H_\alpha\left[\frac{df}{dx}\right]$ given below

$$H_\alpha\left[\frac{df}{dx}\right] = -\frac{1}{2}\,H_\alpha\left(\frac{f}{x}\right) + \,icot\frac{\alpha}{2}\,H_\alpha(fx) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(f) \qquad (8)$$

Now for the second derivative we replace f by fx

As $f \to fx$ so $\frac{df}{dx} = f + x\frac{df}{dx}$

Putting the values in eq (8) we get

$$H_\alpha\left[f + x\,\frac{df}{dx}\right] = -\frac{1}{2}\,H_\alpha(f) + \,icot\frac{\alpha}{2}\,H_\alpha(fx^2) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx)$$

$$H_\alpha\left[x\,\frac{df}{dx}\right] = -\frac{3}{2}\,H_\alpha(f) + \,icot\frac{\alpha}{2}\,H_\alpha(fx^2) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx) \qquad (9)$$

Now replace f by $\frac{f}{x}$, as $f \to \frac{f}{x}$ so $\frac{df}{dx} = \frac{1}{x}\frac{df}{dx} - \frac{f}{x^2}$

Putting the values in eq (8) we get

$$H_\alpha\left[\frac{1}{x}\frac{df}{dx}-\frac{f}{x^2}\right]=-\frac{1}{2}\,H_\alpha\left(\frac{f}{x^2}\right)+\ icot\frac{\alpha}{2}\,H_\alpha(f)-\frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x}\right)$$

$$H_\alpha\left[\frac{1}{x}\frac{df}{dx}\right]=\frac{1}{2}\,H_\alpha\left(\frac{f}{x^2}\right)+\ icot\frac{\alpha}{2}\,H_\alpha(f)-\frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x}\right) \qquad (10)$$

Now replace f by $\frac{df}{dx}$ in equation (8) we get

$$H_\alpha\left[\frac{d^2f}{dx^2}\right]=-\frac{1}{2}\,H_\alpha\left(\frac{1}{x}\frac{df}{dx}\right)+\ icot\frac{\alpha}{2}\,H_\alpha\left(x\frac{df}{dx}\right)-\frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{df}{dx}\right) \qquad (11)$$

Now putting the values of $\frac{1}{x}\frac{df}{dx}$, $x\frac{df}{dx}$ and $\frac{df}{dx}$ from equation (10), (9) and (8) in eq (11) we get $H_\alpha\left[\frac{d^2f}{dx^2}\right]=-\frac{1}{2}\left(\frac{1}{2}\,H_\alpha\left(\frac{f}{x^2}\right)+\ icot\frac{\alpha}{2}\,H_\alpha(f)-\frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x}\right)\right)+\ icot\frac{\alpha}{2}\left(-\frac{3}{2}\,H_\alpha(f)+\ icot\frac{\alpha}{2}\,H_\alpha(fx^2)-\frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx)\right)-\frac{y}{sin\frac{\alpha}{2}}\left(-\frac{1}{2}\,H_\alpha\left(\frac{f}{x}\right)+\ icot\frac{\alpha}{2}\,H_\alpha(fx)-\frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(f)\ \right)$

$$H_\alpha\left[\frac{d^2f}{dx^2}\right]=\ -\frac{1}{4}\,H_\alpha\left(\frac{f}{x^2}\right)-\frac{i}{2}\,cot\frac{\alpha}{2}\,H_\alpha(f)+\frac{y}{2sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x}\right)-\frac{3}{2}\,icot\frac{\alpha}{2}H_\alpha(f)-cot^2\frac{\alpha}{2}H_\alpha(fx^2)$$
$$-\frac{iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}}H_\alpha(fx)+\frac{y}{2sin\frac{\alpha}{2}}H_\alpha\left(\frac{f}{x}\right)-\frac{iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}}H_\alpha(fx)+\frac{y^2}{sin^2\frac{\alpha}{2}}H_\alpha(f)$$

Hence

$$H_\alpha\left[\frac{d^2f}{dx^2}\right]=\ -\frac{1}{4}\,H_\alpha\left(\frac{f}{x^2}\right)+\frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x}\right)-\frac{2iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}}H_\alpha(fx)-cot^2\frac{\alpha}{2}H_\alpha(fx^2)$$
$$+\left(\frac{y^2}{sin^2\frac{\alpha}{2}}-2icot\frac{\alpha}{2}\right)H_\alpha(f) \qquad (12)$$

Results eqns (8) and (12) are similar as in [8], we extended up to higher derivatives

2.2 Differentiation property of fractional Hankel transform of f involing higher order derivatives

For the third derivative replace f by $\frac{df}{dx}$ in equation (12) we get

$$H_\alpha\left[\frac{d^3f}{dx^3}\right] = -\frac{1}{4}\,H_\alpha\left(\frac{1}{x^2}\frac{df}{dx}\right) + \frac{y}{\sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{1}{x}\frac{df}{dx}\right) - \frac{2iy\cot\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}H_\alpha\left(x\frac{df}{dx}\right) - \cot^2\frac{\alpha}{2}H_\alpha\left(x^2\frac{df}{dx}\right)$$

$$+\left(\frac{y^2}{\sin^2\frac{\alpha}{2}} - 2i\cot\frac{\alpha}{2}\right)H_\alpha\left(\frac{df}{dx}\right) \qquad (13)$$

Now replace f by fx^2 in eq (8)

$f \to fx^2$ so $\frac{df}{dx} = x^2\frac{df}{dx} + 2xf$

Hence

$$H_\alpha\left[x^2\frac{df}{dx} + 2xf\right] = -\frac{1}{2}\,H_\alpha(fx) + \,i\cot\frac{\alpha}{2}\,H_\alpha(fx^3) - \frac{y}{\sin\frac{\alpha}{2}}\,H_\alpha(fx^2)$$

$$H_\alpha\left[x^2\frac{df}{dx}\right] = -\frac{1}{2}\,H_\alpha(fx) - 2H_\alpha(fx) + \,i\cot\frac{\alpha}{2}\,H_\alpha(fx^3) - \frac{y}{\sin\frac{\alpha}{2}}\,H_\alpha(fx^2)$$

$$H_\alpha\left[x^2\frac{df}{dx}\right] = -\frac{5}{2}\,H_\alpha(fx) + \,i\cot\frac{\alpha}{2}\,H_\alpha(fx^3) - \frac{y}{\sin\frac{\alpha}{2}}\,H_\alpha(fx^2) \qquad (14)$$

Now replace f by $\frac{f}{x^2}$ in eqn (8)

$f \to \frac{f}{x^2}$ $\qquad \frac{df}{dx} = \frac{1}{x^2}\frac{df}{dx} - \frac{2f}{x^3}$

Hence

$$H_\alpha\left[\frac{1}{x^2}\frac{df}{dx} - \frac{2f}{x^3}\right] = -\frac{1}{2}\,H_\alpha\left(\frac{f}{x^3}\right) + \,i\cot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x}\right) - \frac{y}{\sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^2}\right)$$

$$H_\alpha\left[\frac{1}{x^2}\frac{df}{dx}\right] = -\frac{1}{2}\,H_\alpha\left(\frac{f}{x^3}\right) + 2H_\alpha\left(\frac{f}{x^3}\right) + \,i\cot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x}\right) - \frac{y}{\sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^2}\right)$$

$$H_\alpha\left[\frac{1}{x^2}\frac{df}{dx}\right] = \frac{3}{2}\,H_\alpha\left(\frac{f}{x^3}\right) + \,i\cot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x}\right) - \frac{y}{\sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^2}\right) \qquad (15)$$

Now putting the values from (8), (9), (10), (14) and (15) in eq (13) we get

$$H_\alpha\left[\frac{d^3f}{dx^3}\right] = -\frac{1}{4}\left[\frac{3}{2}\,H_\alpha\left(\frac{f}{x^3}\right) + \;icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^2}\right)\right]$$

$$+\frac{y}{sin\frac{\alpha}{2}}\left[\frac{1}{2}\,H_\alpha\left(\frac{f}{x^2}\right) + \;icot\frac{\alpha}{2}\,H_\alpha(f) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x}\right)\right]$$

$$-\frac{2iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\left[-\frac{3}{2}\,H_\alpha(f) + \;icot\frac{\alpha}{2}\,H_\alpha(fx^2) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx)\right]$$

$$-cot^2\frac{\alpha}{2}\left[-\frac{5}{2}\,H_\alpha(fx) + \;icot\frac{\alpha}{2}\,H_\alpha(fx^3) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx^2)\right]$$

$$+\left(\frac{y^2}{sin^2\frac{\alpha}{2}} - 2icot\frac{\alpha}{2}\right)\left[-\frac{1}{2}\,H_\alpha\left(\frac{f}{x}\right) + \;icot\frac{\alpha}{2}\,H_\alpha(fx) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(f)\right]$$

$$H_\alpha\left[\frac{d^3f}{dx^3}\right] = -\frac{3}{8}H_\alpha\left(\frac{f}{x^3}\right) + \left[\frac{y}{4sin\frac{\alpha}{2}} + \frac{y}{2sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^2}\right) + \left[-\frac{icot\frac{\alpha}{2}}{4} - \frac{y^2}{sin^2\frac{\alpha}{2}} - \frac{y^2}{2sin^2\frac{\alpha}{2}} + icot\frac{\alpha}{2}\right]H_\alpha\left(\frac{f}{x}\right)$$

$$+\left[\frac{iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}} + \frac{3iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}} - \frac{y^3}{sin^3\frac{\alpha}{2}} + \frac{2iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(f)$$

$$+\left[\frac{2iy^2cot\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}} + \frac{5}{2}cot^2\frac{\alpha}{2} + \frac{iy^2cot\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}} + 2cot^2\frac{\alpha}{2}\right]H_\alpha(fx) + \left[\frac{2ycot^2\frac{\alpha}{2}}{sin\frac{\alpha}{2}} + \frac{ycot^2\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(fx^2)$$

$$-icot^3\frac{\alpha}{2}H_\alpha(fx^3)$$

$$H_\alpha\left[\frac{d^3f}{dx^3}\right] = -\frac{3}{8}H_\alpha\left(\frac{f}{x^3}\right) + \left[\frac{3y}{4sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^2}\right) + \left[\frac{3icot\frac{\alpha}{2}}{4} - \frac{3y^2}{2sin^2\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x}\right) + \left[\frac{6iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}} - \frac{y^3}{sin^3\frac{\alpha}{2}}\right]H_\alpha(f)$$

$$+\left[\frac{3iy^2cot\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}} + \frac{9}{2}cot^2\frac{\alpha}{2}\right]H_\alpha(fx) + \left[\frac{3ycot^2\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(fx^2)$$

$$-icot^3\frac{\alpha}{2}H_\alpha(fx^3) \qquad (16)$$

2.3 Differentiaon property involving fourth derivative

For the fourth derivative we replace f by $\frac{df}{dx}$ in eq (16)

$$H_\alpha\left[\frac{d^4f}{dx^4}\right] = -\frac{3}{8}H_\alpha\left(\frac{1}{x^3}\frac{df}{dx}\right)+\left[\frac{3y}{4sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{1}{x^2}\frac{df}{dx}\right)+\left[\frac{3icot\frac{\alpha}{2}}{4}-\frac{3y^2}{2sin^2\frac{\alpha}{2}}\right]H_\alpha\left(\frac{1}{x}\frac{df}{dx}\right)$$
$$+\left[\frac{6iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}}-\frac{y^3}{sin^3\frac{\alpha}{2}}\right]H_\alpha\left(\frac{df}{dx}\right)+\left[\frac{3iy^2cot\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}+\frac{9}{2}cot^2\frac{\alpha}{2}\right]H_\alpha\left(x\frac{df}{dx}\right)$$
$$+\left[\frac{3ycot^2\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha\left(x^2\frac{df}{dx}\right)-icot^3\frac{\alpha}{2}H_\alpha\left(x^3\frac{df}{dx}\right) \qquad (17)$$

Now replace f by fx^3 and f by $\frac{f}{x^3}$ in eq (8) respectively

$f \rightarrow fx^3$ so $\frac{df}{dx} = x^3\frac{df}{dx} + 3x^2f$

$$H_\alpha\left[x^3\frac{df}{dx}+3x^2f\right] = -\frac{1}{2}H_\alpha(fx^2) + icot\frac{\alpha}{2}H_\alpha(fx^4) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha(fx^3)$$

$$H_\alpha\left[x^3\frac{df}{dx}\right] = -\frac{1}{2}H_\alpha(fx^2) - 3H_\alpha(fx^2) + icot\frac{\alpha}{2}H_\alpha(fx^4) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha(fx^3)$$

$$H_\alpha\left[x^3\frac{df}{dx}\right] = -\frac{7}{2}H_\alpha(fx^2) + icot\frac{\alpha}{2}H_\alpha(fx^4) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha(fx^3) \qquad (18)$$

For the case $f \rightarrow \frac{f}{x^3}$ $\frac{df}{dx} = \frac{1}{x^3}\frac{df}{dx} - \frac{3f}{x^4}$

$$H_\alpha\left[\frac{1}{x^3}\frac{df}{dx} - \frac{3f}{x^4}\right] = -\frac{1}{2}H_\alpha\left(\frac{f}{x^4}\right) + icot\frac{\alpha}{2}H_\alpha\left(\frac{f}{x^2}\right) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha\left(\frac{f}{x^3}\right)$$

$$H_\alpha\left[\frac{1}{x^3}\frac{df}{dx}\right] = -\frac{1}{2}H_\alpha\left(\frac{f}{x^4}\right) + 3H_\alpha\left(\frac{f}{x^4}\right) + icot\frac{\alpha}{2}H_\alpha\left(\frac{f}{x^2}\right) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha\left(\frac{f}{x^3}\right)$$

$$H_\alpha\left[\frac{1}{x^3}\frac{df}{dx}\right] = \frac{5}{2}H_\alpha\left(\frac{f}{x^4}\right) + icot\frac{\alpha}{2}H_\alpha\left(\frac{f}{x^2}\right) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha\left(\frac{f}{x^3}\right) \qquad (19)$$

Now putting the values from(8), (9),(10),(14) , (15),(18) and (19) in eq (17) we get

$$H_\alpha\left[\frac{d^4f}{dx^4}\right] = -\frac{3}{8}\left[\frac{5}{2}\,H_\alpha\left(\frac{f}{x^4}\right) + icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x^2}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^3}\right)\right]$$

$$+\left[\frac{3y}{4sin\frac{\alpha}{2}}\right]\left[\frac{3}{2}\,H_\alpha\left(\frac{f}{x^3}\right) + icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^2}\right)\right]$$

$$+\left[\frac{3icot\frac{\alpha}{2}}{4} - \frac{3y^2}{2sin^2\frac{\alpha}{2}}\right]\left[\frac{1}{2}\,H_\alpha\left(\frac{f}{x^2}\right) + icot\frac{\alpha}{2}\,H_\alpha(f) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x}\right)\right]$$

$$+\left[\frac{6iycot\frac{\alpha}{2}}{sin\frac{\alpha}{2}} - \frac{y^3}{sin^3\frac{\alpha}{2}}\right]\left[-\frac{1}{2}\,H_\alpha\left(\frac{f}{x}\right) + icot\frac{\alpha}{2}\,H_\alpha(fx) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(f)\right]$$

$$+\left[\frac{3iy^2cot\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}} + \frac{9}{2}cot^2\frac{\alpha}{2}\right]\left[-\frac{3}{2}\,H_\alpha(f) + icot\frac{\alpha}{2}\,H_\alpha(fx^2) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx)\right]$$

$$+\left[\frac{3ycot^2\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]\left[-\frac{5}{2}\,H_\alpha(fx) + icot\frac{\alpha}{2}\,H_\alpha(fx^3) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx^2)\right]$$

$$-\,icot^3\frac{\alpha}{2}\left[-\frac{7}{2}H_\alpha(fx^2) + icot\frac{\alpha}{2}\,H_\alpha(fx^4) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx^3)\right]$$

$$H_\alpha\left[\frac{d^4f}{dx^4}\right] = -\frac{15}{16}H_\alpha\left(\frac{f}{x^4}\right) + \left[\frac{3y}{8sin\frac{\alpha}{2}} + \frac{9y}{8sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^3}\right)$$

$$+\left[-\frac{3icot\frac{\alpha}{2}}{8} - \frac{3y^2}{4sin^2\frac{\alpha}{2}} + \frac{3icot\frac{\alpha}{2}}{8} - \frac{3y^2}{4sin^2\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^2}\right)$$

$$+\left[\frac{3iycot\frac{\alpha}{2}}{4sin\frac{\alpha}{2}} - \frac{3iycot\frac{\alpha}{2}}{4sin\frac{\alpha}{2}} + \frac{3y^3}{2sin^3\frac{\alpha}{2}} - \frac{6iycot\frac{\alpha}{2}}{2sin\frac{\alpha}{2}} + \frac{y^3}{2sin^3\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x}\right)$$

$$+\left[-\frac{3}{4}cot^2\frac{\alpha}{2} - \frac{3iy^2cot\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}} - \frac{6iy^2cot\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}} + \frac{y^4}{sin^4\frac{\alpha}{2}} - \frac{9iy^2cot\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}} - \frac{27}{4}cot^2\frac{\alpha}{2}\right]H_\alpha(f)$$

$$+\left[-\frac{6ycot^2\frac{\alpha}{2}}{sin\frac{\alpha}{2}} - \frac{iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}} - \frac{3iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}} - \frac{9ycot^2\frac{\alpha}{2}}{2sin\frac{\alpha}{2}} - \frac{15ycot^2\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]H_\alpha(fx)$$

$$+\left[-\frac{3y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}} + \frac{9}{2}icot^3\frac{\alpha}{2} - \frac{3y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}} + \frac{7}{2}icot^3\frac{\alpha}{2}\right]H_\alpha(fx^2)$$

$$+\left[\frac{3iycot^3\frac{\alpha}{2}}{sin\frac{\alpha}{2}} + \frac{iycot^3\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(fx^3) + cot^4\frac{\alpha}{2}H_\alpha(fx^4)$$

$$H_\alpha\left[\frac{d^4f}{dx^4}\right] = -\frac{15}{16}H_\alpha\left(\frac{f}{x^4}\right) + \left[\frac{12y}{8sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^3}\right) - \left[\frac{6y^2}{4sin^2\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^2}\right) + \left[\frac{2y^3}{sin^3\frac{\alpha}{2}} - \frac{6iycot\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x}\right)$$
$$+\left[-\frac{30}{4}cot^2\frac{\alpha}{2} - \frac{24iy^2cot\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}} + \frac{y^4}{sin^4\frac{\alpha}{2}}\right]H_\alpha(f) + \left[-\frac{4iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}} - \frac{36ycot^2\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]H_\alpha(fx)$$
$$+\left[\frac{16}{2}icot^3\frac{\alpha}{2} - \frac{6y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}\right]H_\alpha(fx^2) + \left[\frac{4iycot^3\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(fx^3)$$
$$+ cot^4\frac{\alpha}{2}H_\alpha(fx^4) \qquad (20)$$

2.4 Operational relation involving 5^th derivatives will be established by replacing

f by $\frac{df}{dx}$ in equation (20) and f by fx^4 in eq (8)

we get

$$H_\alpha\left[\frac{d^5f}{dx^5}\right] = -\frac{15}{16}H_\alpha\left(\frac{1}{x^4}\frac{df}{dx}\right) + \left[\frac{12y}{8sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{1}{x^3}\frac{df}{dx}\right) - \left[\frac{6y^2}{4sin^2\frac{\alpha}{2}}\right]H_\alpha\left(\frac{1}{x^2}\frac{df}{dx}\right)$$
$$+\left[\frac{2y^3}{sin^3\frac{\alpha}{2}} - \frac{6iycot\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{1}{x}\frac{df}{dx}\right) + \left[-\frac{30}{4}cot^2\frac{\alpha}{2} - \frac{24iy^2cot\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}} + \frac{y^4}{sin^4\frac{\alpha}{2}}\right]H_\alpha\left(\frac{df}{dx}\right)$$
$$+\left[-\frac{4iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}} - \frac{36ycot^2\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]H_\alpha\left(x\frac{df}{dx}\right) + \left[\frac{16}{2}icot^3\frac{\alpha}{2} - \frac{6y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}\right]H_\alpha\left(x^2\frac{df}{dx}\right)$$
$$+\left[\frac{4iycot^3\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha\left(x^3\frac{df}{dx}\right) + cot^4\frac{\alpha}{2}H_\alpha\left(x^4\frac{df}{dx}\right) \qquad (21)$$

Now replace f by fx^4 in eq (8)

$f \rightarrow fx^4$ so $\frac{df}{dx} = x^4\frac{df}{dx} + 4x^3f$

$$H_\alpha\left[x^4\frac{df}{dx} + 4x^3f\right] = -\frac{1}{2}H_\alpha(fx^3) + icot\frac{\alpha}{2}H_\alpha(fx^5) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha(fx^4)$$

$$H_\alpha\left[x^4\frac{df}{dx}\right] = -\frac{1}{2}H_\alpha(fx^3) - 4H_\alpha(fx^3) + icot\frac{\alpha}{2}H_\alpha(fx^5) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha(fx^4)$$

$$H_\alpha\left[x^4\frac{df}{dx}\right] = -\frac{9}{2}H_\alpha(fx^3) + icot\frac{\alpha}{2}H_\alpha(fx^5) - \frac{y}{sin\frac{\alpha}{2}}H_\alpha(fx^4) \qquad (22)$$

Now replace f by $\frac{f}{x^4}$ in eq (8)

$$f \to \frac{f}{x^4} \qquad \frac{df}{dx} = \frac{1}{x^4}\frac{df}{dx} - \frac{4f}{x^5}$$

$$H_\alpha\left[\frac{1}{x^4}\frac{df}{dx} - \frac{4f}{x^5}\right] = -\frac{1}{2}\,H_\alpha\left(\frac{f}{x^5}\right) + \,icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x^3}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^4}\right)$$

$$H_\alpha\left[\frac{1}{x^4}\frac{df}{dx}\right] = -\frac{1}{2}\,H_\alpha\left(\frac{f}{x^5}\right) + 4H_\alpha\left(\frac{f}{x^5}\right) + \,icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x^3}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^4}\right)$$

$$H_\alpha\left[\frac{1}{x^4}\frac{df}{dx}\right] = \frac{7}{2}\,H_\alpha\left(\frac{f}{x^5}\right) + \,icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x^3}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^4}\right) \qquad (23)$$

Now putting the values from(8), (9),(10),(14) , (15),(18) , (19),(22) and (23) in eq (21) we get

$$\begin{aligned}
H_\alpha\left[\frac{d^5f}{dx^5}\right] = &-\frac{15}{16}\left[\frac{7}{2}\,H_\alpha\left(\frac{f}{x^5}\right) + \,icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x^3}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^4}\right)\right] \\
&+ \left[\frac{12y}{8sin\frac{\alpha}{2}}\right]\left[\frac{5}{2}\,H_\alpha\left(\frac{f}{x^4}\right) + \,icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x^2}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^3}\right)\right] \\
&- \left[\frac{6y^2}{4sin^2\frac{\alpha}{2}}\right]\left[\frac{3}{2}\,H_\alpha\left(\frac{f}{x^3}\right) + \,icot\frac{\alpha}{2}\,H_\alpha\left(\frac{f}{x}\right) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x^2}\right)\right] \\
&+ \left[\frac{2y^3}{sin^3\frac{\alpha}{2}} - \frac{6iycot\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]\left[\frac{1}{2}\,H_\alpha\left(\frac{f}{x^2}\right) + \,icot\frac{\alpha}{2}\,H_\alpha(f) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha\left(\frac{f}{x}\right)\right] \\
&+ \left[-\frac{30}{4}cot^2\frac{\alpha}{2} - \frac{24iy^2cot\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}} + \frac{y^4}{sin^4\frac{\alpha}{2}}\right]\left[-\frac{1}{2}\,H_\alpha\left(\frac{f}{x}\right) + \,icot\frac{\alpha}{2}\,H_\alpha(fx) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(f)\right] \\
&+ \left[-\frac{4iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}} - \frac{36ycot^2\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]\left[-\frac{3}{2}\,H_\alpha(f) + \,icot\frac{\alpha}{2}\,H_\alpha(fx^2) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx)\right] \\
&+ \left[\frac{16}{2}icot^3\frac{\alpha}{2} - \frac{6y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}\right]\left[-\frac{5}{2}\,H_\alpha(fx) + \,icot\frac{\alpha}{2}\,H_\alpha(fx^3) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx^2)\right] \\
&+ \left[\frac{4iycot^3\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]\left[-\frac{7}{2}H_\alpha(fx^2) + \,icot\frac{\alpha}{2}\,H_\alpha(fx^4) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx^3)\right] \\
&+ cot^4\frac{\alpha}{2}\left[-\frac{9}{2}\,H_\alpha(fx^3) + \,icot\frac{\alpha}{2}\,H_\alpha(fx^5) - \frac{y}{sin\frac{\alpha}{2}}\,H_\alpha(fx^4)\right]
\end{aligned}$$

$$H_\alpha\left[\frac{d^5f}{dx^5}\right] = -\frac{105}{32}H_\alpha\left(\frac{f}{x^5}\right)+\left[\frac{15y}{16sin\frac{\alpha}{2}}+\frac{15y}{4sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^4}\right)+\left[-\frac{15}{16}icot\frac{\alpha}{2}-\frac{3y^2}{2sin^2\frac{\alpha}{2}}-\frac{9y^2}{4sin^2\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^3}\right)$$
$$+\left[\frac{3iycot\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}+\frac{3y^3}{2sin^3\frac{\alpha}{2}}+\frac{y^3}{sin^3\frac{\alpha}{2}}-\frac{3iycot\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^2}\right)$$
$$+\left[\frac{-3iy^2cot\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}}-\frac{2y^4}{sin^4\frac{\alpha}{2}}+\frac{3iy^2cot\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}+\frac{15}{4}cot^2\frac{\alpha}{2}+\frac{6iy^2cot\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}-\frac{y^4}{2sin^4\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x}\right)$$
$$+\left[\frac{2iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}}+\frac{15ycot^2\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}+\frac{12iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}}-\frac{y^5}{sin^5\frac{\alpha}{2}}+\frac{6iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}}+\frac{27ycot^2\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(f)$$
$$+\left[-\frac{15}{2}icot^3\frac{\alpha}{2}-\frac{12y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}+\frac{iy^4cot\frac{\alpha}{2}}{sin^4\frac{\alpha}{2}}+\frac{4iy^4cot\frac{\alpha}{2}}{sin^4\frac{\alpha}{2}}+\frac{18y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}-20icot^3\frac{\alpha}{2}\right.$$
$$\left.+\frac{3y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}\right]H_\alpha(fx)$$
$$+\left[\frac{4y^3cot^2\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}}-\frac{18iycot^3\frac{\alpha}{2}}{sin\frac{\alpha}{2}}-\frac{8iycot^3\frac{\alpha}{2}}{sin\frac{\alpha}{2}}+\frac{6y^3cot^2\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}}-\frac{28iycot^3\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right]H_\alpha(fx^2)$$
$$+\left[-8cot^4\frac{\alpha}{2}-\frac{6iy^2cot^3\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}-\frac{4iy^2cot^3\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}-\frac{9}{2}cot^4\frac{\alpha}{2}\right]H_\alpha(fx^3)$$
$$+\left[-\frac{4ycot^4\frac{\alpha}{2}}{sin\frac{\alpha}{2}}-\frac{ycot^4\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(fx^4)+icot^5\frac{\alpha}{2}\ H_\alpha(fx^5)$$

$$H_\alpha\left[\frac{d^5f}{dx^5}\right] = -\frac{105}{32}H_\alpha\left(\frac{f}{x^5}\right)+\left[\frac{75y}{16sin\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^4}\right)+\left[-\frac{15}{16}icot\frac{\alpha}{2}-\frac{15y^2}{4sin^2\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^3}\right)+\left[\frac{5y^3}{2sin^3\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x^2}\right)$$
$$+\left[\frac{15}{4}cot^2\frac{\alpha}{2}+\frac{15iy^2cot\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}}-\frac{5y^4}{2sin^4\frac{\alpha}{2}}\right]H_\alpha\left(\frac{f}{x}\right)$$
$$+\left[\frac{20iy^3cot\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}}+\frac{69ycot^2\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}-\frac{y^5}{sin^5\frac{\alpha}{2}}\right]H_\alpha(f)$$
$$+\left[-\frac{55}{2}icot^3\frac{\alpha}{2}+\frac{33y^2cot^2\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}+\frac{5iy^4cot\frac{\alpha}{2}}{sin^4\frac{\alpha}{2}}\right]H_\alpha(fx)$$
$$+\left[\frac{10y^3cot^2\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}}-\frac{40iycot^3\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(fx^2)+\left[-\frac{25}{2}cot^4\frac{\alpha}{2}-\frac{10iy^2cot^3\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}\right]H_\alpha(fx^3)$$
$$+\left[-\frac{5ycot^4\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right]H_\alpha(fx^4)+icot^5\frac{\alpha}{2}\ H_\alpha(fx^5) \qquad (24)$$

2.5 Differentiaon property involving 6^{th} derivative

Replace f by $\frac{df}{dx}$ in equation (24) we get and substitution of fx^5 and f by $\frac{f}{x^5}$ in eq (8)

One can established with the similar procedure

$$H_\alpha\left[\frac{d^6 f}{dx^6}\right] = -\frac{945}{64} H_\alpha\left(\frac{f}{x^6}\right) + \left[\frac{630y}{32sin\frac{\alpha}{2}}\right] H_\alpha\left(\frac{f}{x^5}\right) + \left[-\frac{180}{32} icot\frac{\alpha}{2} - \frac{225y^2}{16sin^2\frac{\alpha}{2}}\right] H_\alpha\left(\frac{f}{x^4}\right)$$
$$+\left[\frac{90iycot\frac{\alpha}{2}}{16sin\frac{\alpha}{2}} + \frac{15y^3}{4sin^3\frac{\alpha}{2}}\right] H_\alpha\left(\frac{f}{x^3}\right) + \left[\frac{45}{16}cot^2\frac{\alpha}{2} - \frac{5y^4}{4sin^4\frac{\alpha}{2}}\right] H_\alpha\left(\frac{f}{x^2}\right)$$
$$+\left[\frac{20iy^3cot\frac{\alpha}{2}}{2sin^3\frac{\alpha}{2}} - \frac{15ycot^2\frac{\alpha}{2}}{4sin\frac{\alpha}{2}} - \frac{5y^5}{2sin^5\frac{\alpha}{2}}\right] H_\alpha\left(\frac{f}{x}\right)$$
$$+\left[\frac{180}{4}icot^3\frac{\alpha}{2} - \frac{282y^2cot^2\frac{\alpha}{2}}{2sin^2\frac{\alpha}{2}} - \frac{75iy^4cot\frac{\alpha}{2}}{2sin^4\frac{\alpha}{2}} + \frac{y^6}{sin^6\frac{\alpha}{2}}\right] H_\alpha(f)$$
$$+\left[-\frac{78y^3cot^2\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}} + \frac{324iycot^3\frac{\alpha}{2}}{2sin\frac{\alpha}{2}} - \frac{6iy^5cot\frac{\alpha}{2}}{sin^5\frac{\alpha}{2}}\right] H_\alpha(fx)$$
$$+\left[\frac{285}{4}cot^4\frac{\alpha}{2} + \frac{143iy^2cot^3\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}} - \frac{15y^4cot^2\frac{\alpha}{2}}{sin^4\frac{\alpha}{2}}\right] H_\alpha(fx^2)$$
$$+\left[\frac{20iy^3cot^3\frac{\alpha}{2}}{sin^3\frac{\alpha}{2}} + \frac{150ycot^4\frac{\alpha}{2}}{2sin\frac{\alpha}{2}}\right] H_\alpha(fx^3) + \left[-\frac{36}{2}icot^5\frac{\alpha}{2} + \frac{15y^2cot^4\frac{\alpha}{2}}{sin^2\frac{\alpha}{2}}\right] H_\alpha(fx^4)$$
$$+\left[\frac{-6iycot^5\frac{\alpha}{2}}{sin\frac{\alpha}{2}}\right] H_\alpha(fx^5) - cot^6\frac{\alpha}{2} H_\alpha(fx^6) \qquad (25)$$

Thus the differential property of fractional Hankel transform for any higher order derivatives of a function can be established in a similar way which will be helpful to solve differential equations of any order.

3. Conclusion

We have contributed the mathematical formulation of differential property of Fractional Hankel transform in very easier manner and extended it to higher order derivative of a function which will play a vital role for solving differential equations of higher order differential equations.

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