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Relationship of Bell's Polynomial Matrix and *k*-Fibonacci Matrix

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Abstract

The Bell's polynomial matrix is expressed as B_n, where each of its entry represents the Bell's polynomial number.This Bell's polynomial number functions as an information code of the number of ways in which partitions of a set with certain elements are arranged into several non-empty section blocks. Furthermore, the*k*-Fibonacci matrix is expressed as$F_n(k)$, where each of its entry represents the *k*-Fibonacci number, whose first term is 0, the second term is 1 and the next term depends on a positive integer *k*. This article aims to find a matrix based on the multiplication of the Bell's polynomial matrix and the *k*-Fibonacci matrix. Then from the relationship between the two matrices the matrix Y_n is obtained. The matrix $Y_n$is not commutative from the product of the two matrices, so we get matrixZ_n.Thus, the matrix $Y_n \neq Z_n$, so that the Bell's polynomial matrix relationship can be expressed as$B_n{=}F_n(k)Y_n{=}Z_nF_n(k)$.

Keywords: Bell's Polynomial Number; Bell's Polynomial Matrix; *k*-Fibonacci Number; *k*-Fibonacci Matrix.

1. Introduction

Bell's polynomial numbers are studied by mathematicians since the 19th century and are named after their inventor Eric Temple Bell in 1938[5]. Bell's polynomial numbers are represented by $B_{n,k}$ for each n and *k*elementsof the natural number, starting from $B_{n,1}{=}x_n$and $B_{n,n}{=}x_1^n$ [2, p. 135]. Bell's polynomial numbers can be formed into Bell's polynomial matrix so that each entry of Bell's polynomial matrix is Bell's polynomial number and is represented by B_n [11].

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Reference [1] discuss the relationship of Bell's polynomial numbers and Stirling number of the second kind involving Bell's number, Reference [5] discuss the relationship between repetition of partition vectors for Bell's polynomial number, Reference [6] discusses Bell's polynomial number involving binomial lines, Qi and his colleagues [7] define the relationship between Bell's polynomial number andStirling numbers of the first kind and Stirling numbers of the second kind, Wang and Wang [10] discuss the relationship of the Bell's polynomial matrix by involving the Pascal matrix and the Stirling matrix. Reference [11] also define the relationship of the Bell's polynomial matrix with the Fibonacci matrix, and from the relationship of the two matrices matrix M_n and matrix N_n are obtained, so the Bell's polynomial matrix is defined as $B_n=F_nM_n$ and $B_n=N_nF_n$. The k-Fibonacci number is discovered by [3]. The k-Fibonacci number is denoted by $F_{k,n}$ for each k and n elements of the natural number. The k-Fibonacci number can be formed into a k-Fibonacci matrix and denoted by $F_n(k)$. Reference [4] discuss the k-Fibonacci modulo m, Reference [8] discuss the relationship between the k-Fibonacci sequence and the generalization of the k-Fibonacci sequence, Wahyuni and his colleagues [9] discuss the identity associated with the k-Fibonacci sequence modulo ring Z_6 and R_n. Reference [3] also defines the relationship of the k-Fibonacci matrix with Pascal matrix, and from the relationship of the two matrices $L_n$matrix and Rn matrix are obtained, so the Pascal matrix is defined as $P_n=F_n(k)L_n$ and $P_n=R_nF_n(k)$ for each element n and k natural numbers. This article discusses the Bell's polynomial matrix involving the k-Fibonacci matrix $F_n(k)$ and the inverse k-Fibonacci matrix $F_n^{-1}(k)$which aims to find two matrix forms. Furthermore, by studying the relationship of the Bell's polynomial matrix and the k-Fibonacci matrix, the second part discusses the supporting theory of the Bell's polynomial matrix and the k-Fibonacci matrix, in the third part the results of the relationship between the two Bell's polynomial matrices and the k-Fibonacci matrix, and in the fourth section presents the conclusion of this research.

2. Bell's Polynomial Matrix and *k*-Fibonacci Matrix

Bell polynomial numbers are defined by Comtet [2, p. 133] in Definition 1.

Definition 1 The Bell's polynomialare the polynomials$B_{n,k} = B_{n,k}(x_{1,} x_2, ..., x_{n-k+1})$ in infinite numbers of variables$x_{1,} x_2, ...,$defined by formal double series expansion

$$\Phi=\Phi(t,u) := \exp\left(u\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)=\sum_{n,k\geq 0} B_{n,k}\frac{t^m}{n!}u^k,$$

$$= 1 + \sum_{n\geq 1}\frac{t^m}{n!}\left\{\sum_{k=1}^{n} u^n B_{n,k}(x_1 , x_2, ...)\right\},$$

or, the amount is the same as the series expansion

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k}\frac{t^n}{n!}, k=0, 1, 2, \ldots, n.$$

Based on the definition of Bell's polynomial number, Comtet [2, p. 134] gives Theorem 2 for Bell's polynomial number.

Theorem 2 Bell's polynomial numbers have integer coefficients, where the number equals$k$ and the weight equals$n$ expressed by

$$B_{n,k}\left(x_1, x_2, \dots, x_{n-k+1}\right)=\sum \frac{n!}{c_1!\, c_2! \dots c_{n-k+1}!}\left(\frac{x_1}{1!}\right)^{c_1}\left(\frac{x_2}{2!}\right)^{c_2}\dots\left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{c_{n-k+1}}. \qquad (1)$$

The sum is taken over all rows $c_1,\ c_2,\dots,\ c_{n-k+1}$ for nonnegative integers such that up to two conditions the following are met:

(i) $c_1 + c_2 + \cdots + c_{n-k+1}=k$,

(ii) $c_1 + 2c_2 + 3c_3 + \cdots + (n-k+1)c_{n-k+1}=n$.

Then the Bell's polynomial numbers to n is the number

$$B_{n,k}=B_{n,k}\left(x_1, x_2, \dots, x_{n-k+1}\right).$$

Proof. Proof of this theorem can be seen in Comtet [2, p. 134]. ■

The Bell's polynomial matrix is a lower triangular matrix where each entry is a Bell's polynomial number. Wang and Wang [11] provide the Bell's polynomial matrix in Definition 3.

Definition 3 For each natural number n, the Bell's polynomial matrix n x n with each entry being the Bell's polynomial numbers is expressed as

$$(B_n)i,j=[B_{i,j}]\text{for each}i,\ j = 1, 2, 3, \dots, n. \qquad (2)$$

Based on the Bell's polynomial matrix definition, the general form of the Bell's polynomial matrix is stated as follows:

$$B_n=\begin{bmatrix} B_{1,1} & B_{1,2} & B_{1,3} & \cdots & B_{1,j} \\ B_{2,1} & B_{2,2} & B_{2,3} & \cdots & B_{2,j} \\ B_{3,1} & B_{3,2} & B_{3,3} & \cdots & B_{3,j} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{i,1} & B_{i,2} & B_{i,3} & \cdots & B_{i,n} \end{bmatrix}.$$

The next discussion is the k-Fibonacci number. The k-Fibonacci number is defined by Falcon [4] in Definition 4.

Definition 4 For each natural number n and k, the k-Fibonacci number is defined as

$F_{k,n+1}= kF_{k,n} + F_{k,n-1}$ for $n \geq 1$, (3)with initial conditions $F_{k,0}$=0 and $F_{k,1}$=0.

Furthermore, like the Bell's polynomial matrix, the *k*-Fibonacci matrix is also a lower triangular matrix with each entry a *k*-Fibonacci number. Reference [3] gives the definition of the *k*-Fibonacci matrix in Definition 5.

Definition 5 For each naturalnumber *n*, the*k*-Fibonaccimatrix *n*x*n* whose entry is *k*-Fibonacci numbers expressed by$F_n(k)=[f_{i,j}(k)]$, for each$i, j = 1, 2, 3, \dots, n$, are defined as

$$f_{i,j}(k)=\begin{cases} F_{k,i-j+1}, & \text{if } i\geq j, \\ 0, & \text{if } i<j. \end{cases} \quad (4)$$

Based on the definition of the *k*-Fibonacci matrix, the general form of*k*-Fibonacci is stated as follows:

$$F_n(k)=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ F_{k,2} & 1 & 0 & \cdots & 0 \\ F_{k,3} & F_{k,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{k,n} & F_{k,n-1} & F_{k,n-2} & \cdots & 1 \end{bmatrix}.$$

Furthermore, from the general form of the *k*-Fibonacci matrix the *k*-Fibonacci matrix $F_n(k)$is obtained with the main diagonal of 1 and the determinant value (det) of the *k*-Fibonacci matrix $F_n(k)$ is the result of multiplying diagonal entries, so we get $\det(F_n(k))=1$. Because $\det(F_n(k))\neq 0$, the *k*-Fibonacci matrix $F_n(k)$ has an inverse. Reference [3] provides a definition for the inverse *k*-Fibonacci matrix in Definition 6.

Definition 6 Let$F_n^{-1}(k)$be the inverse of the *k*-Fibonacci matrix, for each natural number *n* where each entry of the inverse *k*-Fibonacci matrix$F_n^{-1}(k) = [f'_{i,j}(k)]$, for each$i, j = 1, 2, 3, \dots, n$, is defined as

$$f'_{i,j}(k)=\begin{cases} 1, & \text{if } i=j, \\ -k, & \text{if } i-1=j, \\ -1, & \text{if } i-2=j, \\ 0, & \text{for others.} \end{cases} \quad (5)$$

Because the *k*-Fibonacci matrix$F_n(k)$ has an inverse, then $F_n(k)F_n^{-1}(k)=I_n=F_n^{-1}(k)F_n(k)$ can be applied, so the *k*-Fibonacci matrix is an invertible matrix.Based on the definition of the inverse *k*-Fibonacci matrix, the general form of the inverse *k*-Fibonacci matrix is

$$F_n^{-1}(k)=\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 \\ -k & 1 & 0 & \cdots & \cdots & 0 \\ -1 & -k & 1 & \cdots & \cdots & 0 \\ 0 & -1 & -k & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & 0 & -1 & -k & 1 \end{bmatrix}. \quad (6)$$

Based on equation (6) it can be concluded that for each entry of the inverse *k*-Fibonacci matrix $F_n^{-1}(k)$ applies to each entry pattern in thecolumn$f'_{i,j}(k)$with entries$1, -k, -1$ and 0. This applies to the matrix *n*x*n* where the entry pattern$f'_{i,j}(k)$ does not change.

3. Relationship of Bell's Polynomial Matrix and *k*-Fibonacci Matrix

This section discusses the relationship between the Bell's polynomial matrix and the *k*-Fibonacci matrix. The first relationship of the Bell's polynomial matrix and the *k*-Fibonacci matrix involves the inverse of the *k*-Fibonacci matrix with the Bell's polynomial matrix, so that the matrix is denoted by the matrix Y_n for each of the natural number*n*. The relationship between the two Bell's polynomial matrices and the *k*-Fibonacci matrix by involves the multiplication of the Bell's polynomial matrix and the inverse *k*-Fibonacci matrix, so that a matrix is denoted by the matrix Z_n for every natural number *n*.

3.1. The First Relationship for the Bell Polynomial Matrix and the k-Fibonacci Matrix

Based on the inverse of*k*-Fibonacci matrix in equation (5) and the Bell's polynomial matrix in equation (2), for *n* = 2 multiplying the inverse *k*-Fibonacci matrix $F_2^{-1}(k)$and Bell'spolynomial matrix B_2yields

$$F_2^{-1}(k)B_2=\begin{bmatrix}1 & 0\\ -k & 1\end{bmatrix}\begin{bmatrix}x_1 & 0\\ x_2 & x_1^2\end{bmatrix}.$$

Thus, the product of $F_2^{-1}(k)B_2$ is expressed as a matrixY_2, so a matrix $Y_2$is obtained as follows:

$$Y_2=\begin{bmatrix}x_1 & 0\\ x_2-kx_1 & x_1^2\end{bmatrix}.$$

Next for n = 3 multiplying the inverse of the *k*-Fibonacci matrix $F_3^{-1}(k)$ and the Bell's polynomial matrixB_3gives

$$F_3^{-1}(k)B_3 = \begin{bmatrix}1 & 0 & 0\\ -k & 1 & 0\\ -1 & -k & 1\end{bmatrix}\begin{bmatrix}x_1 & 0 & 0\\ x_2 & x_1^2 & 0\\ x_3 & 3x_1x_2 & x_1^3\end{bmatrix}$$

Thus, the product of $F_3^{-1}(k)B_3$ is expressed as a matrix Y_3, so a matrix Y_3 is obtained as follows:

$$Y_3=\begin{bmatrix}x_1 & 0 & 0\\ x_2-kx_1 & x_1^2 & 0\\ x_3-kx_2-x_1 & 3x_1x_2-kx_1^2 & x_1^3\end{bmatrix}.$$

Next for n = 4 multiplying the inverse of the *k*-Fibonacci matrix $F_4^{-1}(k)$ and the Bell's polynomial matrixB_4produces

$$F_4^{-1}(k)B_4=\begin{bmatrix}1 & 0 & 0 & 0\\ -k & 1 & 0 & 0\\ -1 & -k & 1 & 0\\ 0 & -1 & -k & 1\end{bmatrix}\begin{bmatrix}x_1 & 0 & 0 & 0\\ x_2 & x_1^2 & 0 & 0\\ x_3 & 3x_1x_2 & x_1^3 & 0\\ x_4 & 4x_1x_3+3x_2^2 & 6x_1^2x_2 & x_1^4\end{bmatrix}.$$

Thus, the product of$F_4^{-1}(k)B_4$ is expressed as a matrixY_4, so a matrixY_4 is obtained as follows:

$$Y_4=\begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2-kx_1 & x_1^2 & 0 & 0 \\ x_3-kx_2-x_1 & 3x_1x_2-kx_1^2 & x_1^3 & 0 \\ x_4-kx_3-x_2 & 4x_1x_3+3x_2^2-kx_3x_2x_1 & 6x_1^2x_2-kx_1^3 & x_1^4 \end{bmatrix}.$$

By paying attention to each entry in the matrix Y_4there is a matrix $Y_3$and by paying attention to each entry in the Y_3 matrix there is a Y_2 matrix, for $i\geq j$ all values of the$Y_n$matrix entries are listed as Table 1.

Table 1: Element values for matrix Y_n

MatrixY_n Entry	**MatrixY_n Entry Value**
$y_{1,1}$	$(1)B_{1,1}=B_{1,1}$
$y_{2,2}$	$(1)B_{2,2}=B_{2,2}$
$y_{3,3}$	$(1)B_{3,3}=B_{3,3}$
$y_{4,4}$	$(1)B_{4,4}=B_{4,4}$
$y_{2,1}$	$(-k)B_{1,1}+(1)B_{2,1}=B_{2,1}-kB_{1,1}$
$y_{3,2}$	$(-k)B_{2,2}+(1)B_{3,2}=B_{3,2}-kB_{2,2}$
$y_{4,3}$	$(-k)B_{3,3}+(1)B_{4,3}=B_{4,3}-kB_{3,3}$
$y_{3,1}$	$(-1)B_{1,1}+(-k)B_{2,1}+(1)B_{3,1}=B_{3,1}-kB_{2,1}-B_{1,1}$
$y_{4,2}$	$(-1)B_{2,2}+(-k)B_{3,2}+(1)B_{4,2}=B_{4,2}-kB_{3,2}-B_{2,2}$
$y_{4,1}$	$(0)B_{1,1}+(-1)B_{2,1}+(-k)B_{3,1}+(1)B_{4,1}=B_{4,1}-kB_{3,1}-B_{2,1}-0$
$\vdots$	$\vdots$
$y_{i,j}$	$=B_{i,j}-kB_{i\text{-}1,j}-B_{i\text{-}2,j}$

Then paying attention to each step of Table 1 by paying attention to each matrix entry$y_{i,j}$ and the matrix valueY_n, obtained Definition 7 can be stated.

Definition7 For each natural number *n*, the matrix$Y_n$whose order is *n* x *n* with each entry $Y_n=[y_{i,j}]$, for eachi,j=1, 2, 3, …, *n*,isdefined as

$$y_{i,j}=B_{i,j}-B_{i\text{-}1,j}-B_{i\text{-}2,j}. \qquad (7)$$

From equation (7) we get the value of $y_{1,1}=x_1$, $y_{1,j}=0$ for each $j\geq 2$, $y_{2,1}=x_2-kx_1$, $y_{2,2}=x_1^2$, , $y_{2,j}=0$for each$j\geq 3$, and for each $i,j\geq 2$,$y_{i,j}=B_{i,j}-B_{i\text{-}1,j}-B_{i-2,j}$. Based on the definition of the matrix Y_n in equation (7), the Bell's polynomial matrix in equation (2) and the *k*-Fibonacci matrix in equation (4) Theorem 8 can be stated.

Theorem8 Bell's polynomial matrix $B_n$can be grouped into $B_n=F_n(k)Y_n$ **.**

Proof. Since the k-Fibonacci matrix has an inverse, it can be proven that

$Y_n = F_n^{-1}(k)B_n$. (8)

Let $F_n^{-1}(k)$ be the inverse of the k-Fibonacci matrix defined in equation (5) obtained by the main diagonal of the inverse matrix$F_n^{-1}(k)$ =1. Then, based on the Bell's polynomial matrix in equation (2) the main diagonalmatrix $B_n = x_1^n$ is obtained. By paying attention to the left-hand side of equation (7), if $i = j$then $y_{i,j} = x_1^i$ and if$i > j$then$y_{i,j} = 0$. So for $i > 2$ based on equation (5) and equation (2) the values of $y_{i,j}$ can be evaluated as follows:

$$y_{i,j} = f'_{i,i}(k)B_{i,j} + f'_{i,i-1}(k)B_{i-1,j} + f'_{i,i-2}(k)B_{i-2,j} + f'_{i,i-3}(k)B_{i-3,j} + \cdots + f'_{i,n}(k)B_{n,j},$$

$$y_{i,j} = \sum_{k=1}^{n} f'_{i,k}(k)B_{k,j}.$$

So $F_n^{-1}(k)B_n = Y_n$ is obtained. Furthermore, from the definition of the matrix Y_n in equation (7), the Bell's polynomial matrix B_n in equation (2) and the k-Fibonacci matrix in equation (4), $B_n = F_n(k)$. ■

3.2. The Second Relationship for the Bell's Polynomial Matrix and the k-Fibonacci Matrix

Based on the Bell's polynomial matrix in equation (2) and the inverseof$k$-Fibonacci matrix in equation (5), for n = 2 multiplying the Bell's polynomial matrix B_2 and the inverse k-Fibonacci matrix $F_2^{-1}(k)$yields

$$B_2F_2^{-1}(k) = \begin{bmatrix} x_1 & 0 \\ x_2 & x_1^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}.$$

Thus, the product of $B_2F_2^{-1}(k)$ is expressed as a matrixZ_2, so matrix$Z_2$is obtained as follows:

$$Z_2 = \begin{bmatrix} x_1 & 0 \\ x_2 - kx_1^2 & x_1^2 \end{bmatrix}.$$

Furthermore, for n = 3 multiplying the Bell's polynomial matrix B_3 and inverse k-Fibonacci matrix $F_3^{-1}(k)$ gives

$$B_3F_3^{-1}(k) = \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & x_1^2 & 0 \\ x_3 & 3x_1x_2 & x_1^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ -1 & -k & 1 \end{bmatrix}.$$

Thus, the product of $B_3F_3^{-1}(k)$ is expressed as a matrix 3, so matrix Z_3 is obtained as follows:

$$Z_4 = \begin{bmatrix} x_1 & 0 & 0 \\ x_2 - kx_1^2 & x_1^2 & 0 \\ x_3 - 3kx_1x_2 - x_1^3 & 3x_1x_2 - kx_1^3 & x_1^3 \end{bmatrix}.$$

Furthermore, for n = 4 by multiplying the Bell's polynomial matrix $B_4$and the inverse k-Fibonacci matrix $F_4^{-1}(k)$

produces

$$B_4F_4^{-1}(k)=\begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & x_1^2 & 0 & 0 \\ x_3 & 3x_1x_2 & x_1^3 & 0 \\ x_4 & 4x_1x_3+3x_2^2 & 6x_1^2x_2 & x_1^4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ -k & 1 & 0 & 0 \\ -1 & -k & 1 & 0 \\ 0 & -1 & -k & 1 \end{bmatrix}.$$

Thus, the product of $B_4F_4^{-1}(k)$ is expressed as a matrixZ_4, so a matrixZ_4 is obtained as follows:

$$Z_4=\begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2-kx_1^2 & x_1^2 & 0 & 0 \\ x_3-3kx_1x_2-x_1^3 & 3x_1x_2-kx_1^3 & x_1^3 & 0 \\ x_4-4kx_1x_3-3kx_2^2-6x_1^2x_2 & 4x_1x_3+3x_2^2-6kx_1^2-x_1^4 & 6x_1^2x_2-kx_1^4 & x_1^4 \end{bmatrix}.$$

By paying attention to each entry in the matrix Z_4there is a matrixZ_3 and by paying attention to each entry in the matrixZ_3 there is a matrixZ_2, for $i\geq j$all values of thematrixZ_n entries are listed as Table 2.

Table 2: Element values for matrixZ_n

MatrixZ_n Entry	MatrixZ_n Entry Value
$z_{1,1}$	$(1)B_{1,1}=B_{1,1}$
$z_{2,2}$	$(1)B_{2,2}=B_{2,2}$
$z_{3,3}$	$(1)B_{3,3}=B_{3,3}$
$z_{4,4}$	$(1)B_{4,4}=B_{4,4}$
$z_{2,1}$	$(1)B_{2,1}+(-k)B_{2,2}=B_{2,1}-kB_{2,2}$
$z_{3,2}$	$(1)B_{3,2}+(-k)B_{3,3}=B_{3,2}-kB_{3,3}$
$z_{4,3}$	$(1)B_{4,3}+(-k)B_{4,4}=B_{4,3}-kB_{4,4}$
$z_{3,1}$	$(1)B_{3,1}+(-k)B_{3,2}+(-k)B_{3,3}=B_{3,1}-kB_{3,2}-B_{3,3}$
$z_{4,2}$	$(1)B_{4,2}+(-k)B_{4,3}+(-1)B_{4,4}=B_{4,2}-kB_{4,3}-B_{4,4}$
$z_{4,1}$	$(1)B_{4,1}+(-k)B_{4,2}+(-1)B_{4,3}+(0)B_{4,4}=B_{4,1}-kB_{4,2}-B_{4,3}-0$
⋮	⋮
$z_{i,j}$	$=B_{i,j}-kB_{i,j+1}-B_{i,j+2}$

Furthermore, by considering each step of Table 3.2 by paying attention to each entry of the matrix $z_{i,j}$ and the value of the matrixZ_n, Definition 9 can be stated.

Definition9 For each natural number n, the matrix Z_n whose order is$n$xn with each entry $Z_n=[z_{i,j}]$, for each i,j=1, 2, 3, …, n, is defined as

$$z_{i,j}=B_{i,j}-B_{i,j+1}-B_{i,j+2}. \quad (9)$$

From equation (9) we get the value $z_{1,1}=x_1$, $z_{1,j}=0$ for each $j\geq 2$, $z_{2,1}=x_2 - kx_1^2$, $z_{2,2}=x_1^2$, and $z_{2,j}=0$ for each $j\geq 3$, and for each $i,j\geq 2$, $z_{i,j}=B_{i,j} - B_{i,j+1} - B_{i,j+2}$. Based on the definition of the matrix Z_n in equation (9), the Bell's polynomial matrix in equation (2) and the k-Fibonacci matrix in equation (4) Theorem 10 can be stated.

Theorem10 The Bell's polynomial matrix can be grouped into $B_n=Z_nF_n(k)$.

Proof. Because the k-Fibonacci matrix has an inverse, it will be proven that

$$Z_n = B_nF_n^{-1}(k).$$

Let $F_n^{-1}(k)$ be the inverse of the k-Fibonacci matrix defined in equation (5) obtained by the main diagonal of the inverse matrix $F_n^{-1}(k) = 1$. Furthermore, based on the Bell's polynomial matrix in equation (2) the main diagonal matrix $B_n = x_1^n$ is obtained. By paying attention to the left-hand side of equation (9), if $i = j$ then $z_{i,j} = x_1^i$ and $i > j$ then $z_{i,j} = 0$. So for $i > 2$ based on equation (2) and equation (5) the values of $z_{i,j}$ can be evaluated as follows:

$$z_{i,j} = B_{i,j}f'_{j,j}(k) + B_{i,j+1}f'_{j+1,j}(k) + B_{i,j+2}f'_{j+2,j}(k) + B_{i,j+3}f'_{j+3,j}(k) + \cdots + B_{i,n}f'_{n,j}(k),$$

$$z_{i,j} = \sum_{k=1}^{n} B_{i,k}f'_{k,j}(k).$$

So $B_nF_n^{-1}(k) = Z_n$ is obtained. Furthermore, from the definition of the matrix Z_n in equation (9), the Bell's polynomial matrix in equation (2) and the k-Fibonacci matrix in equation (4), $B_n = Z_nF_n(k)$. ■

4. Conclusion

This paper discusses the relationship between the Bell's polynomial matrix and the k-Fibonacci matrix. Then, from the relationship between the two matrices, two matrix equations are obtained. The first equation is denoted by Y_n resulting from the multiplication of the matrix $F_n^{-1}(k)B_n$ and the second is denoted by Z_n resulting from the matrix product $B_n F_n^{-1}(k)$. Furthermore, using matrix Y_n it is showed that the Bell's polynomial matrix B_n can be expressed by $B_n = Y_n F_n^{-1}(k)$ and using matrix Z_n shows that the Bell's polynomial matrix B_n can be expressed as $B_n = F_n^{-1}(k)Z_n$. For further research, it can be carried out a research on the relationship of Bell's polynomial matrix and k-Tribonacci matrix and the relationship of Bell's polynomial matrix and k-Stirling matrix.

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