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Invariant Variational Problems

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Abstract

This research is the study of conservation laws using Lie algebra of the symmetry Lie group. In particular, we consider conservation laws for invariant variational problems based on Noether theorem. We have provided a formula of conservation laws in terms of the Lie algebra of symmetry Lie group.

Keywords: conservation laws; Lie algebra; Noether theorem; Invariant Variational problems.

1. Introduction

Lie groups and their Lie algebras are essential tools in the study of general mathematical fields. These include partial differential equations, physical fields and their classification, homogenous spaces, symmetric spaces and differential geometry in general. Conservation Laws and symmetries have always been of considerable interest in science. They are important in the formulation and investigation of many mathematical models. They were used e.g. for proving global existence theorems [1–3] in problems of stability [4,5] in elasticity for studying cracks and dislocations [6,7] in astrophysics [8–10], in designing new radio antennas' [11] and so on. In this paper we consider the invariant variational problems which are invariant under the continuous group according to Lie theory and Noether theorem for symmetry Lie groups

2. Preliminary

2.1. Lie group G is differentiable manifold which is also a group such that the product and inverse operations are differentiable

I. $M: G \times G \to G, M(x, y) = x\,y$ is differentiable

II. $I: G \to G, I(x) = x^{-1}$ is differentiable

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(A Lie Group G is an abstract group and smooth n-dimensional manifold so that multiplication and inverse are smooth).

2.2. Lie algebra over R or C is a vector space R with askew-symmetric R-bilinear form (Lie bracket) [,]

$V\times V \to V$ which satisfies Jacobian identity. We relate the two via the so called left invariant vector fields: we use the standard notation

$L_g: G \to G, h \to g \ and \ R_g: G \ \to G, h \to gh$

we define a vector field X on a Lie Group G to be left invariant if

$d(L_g)h\big(X(h)\big) = X(gh)$ for all $g, h \in$ G or for short $(L_g)(X) = X$.

Geometrically the left invariant vector field can be identified with the tangent space to G at e. these are the infinitesimal generators of the group G.

2.3. Concept of invariant variational problems

In mathematics an invariant property, hold by a class of mathematical objects, which remain unchanged when transformations of certain type are applied to the objects. Consider the Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial q^\alpha} - D_t\left(\frac{\partial \mathcal{L}}{\partial v^\alpha}\right) = 0,\ \alpha = 1, \dots, s. \qquad (2.1)$$

They give a condition for q(t) to provided extremum of the integral

$$\int_{t_1}^{t_2} \mathcal{L}(t, q, v)dt \qquad (2.2)$$

A variation of integral (2.2) is invariant under the group G if

$$\int_{\bar{V}} \mathcal{L}\big(\bar{x}, \bar{u}, \bar{u}_{(1)} \dots, \bar{u}_{(p)}\big)dx = \int_V \mathcal{L}(x, u, u_{(1)}, \dots, u_{(p)})dx. \qquad (2.3)$$

The invariance condition is given by following lemma

Lemma2.1 An integral

$$\int_V \mathcal{L}\big(x, u, u_{(1)}, \dots, u_{(p)}\big)dx \qquad (2.4)$$

Is invariant under the group G if and only if

$$X(\mathcal{L}) + \mathcal{L}\mathrm{D_i}(\xi^{\mathrm{i}}) = 0. \qquad (2.5)$$

Here X is infinitesimal generator of the group G, it has the form

$$X = \xi^{i}(x,u)\frac{\partial}{\partial x^{i}} + \eta^{i}(x,u)\frac{\partial}{\partial u^{\alpha}}. \qquad (2.6)$$

And X in a prolonged version of the generator (2.6) is

$$\mathrm{X} = \xi^{i}\frac{\partial}{\partial_{\mathrm{x}^{i}}} + \eta^{\propto}\frac{\partial}{\partial \mathrm{u}^{\propto}} + \varsigma_{i}^{\propto}\frac{\partial}{\partial \mathrm{u}_{i}^{\propto}} + \cdots + \varsigma_{i_1,\dots,i_s}^{\propto}\frac{\partial}{\partial \mathrm{u}_{i_1,i_s}^{\propto}} + \cdots \qquad (2.7)$$

where

$$\varsigma_{i}^{\propto} = \mathrm{D}_{i}\left(\eta^{\propto} - \xi^{i}\mathrm{u}_{\mathrm{j}}^{\propto}\right) + \xi^{i}\mathrm{u}_{\mathrm{j}i}^{\propto},$$

$$\varsigma_{i_1,\dots,i_s}^{\propto} = D_{i_1}\dots D_{i_s}\left(\eta^{\propto} - \xi^{i}\mathrm{u}_{\mathrm{j}}^{\propto}\right) + \xi^{i}\mathrm{u}_{\mathrm{j}i_1\dots i_s}^{\propto}.$$

2. Variational problems

In order to derive the general solution of Noether's variational problem, we begin by considering the variation in the action arising from a variation of the both dependent and the independent variables:

$$\delta S = \int_{R'} \{\mathcal{L}(\varphi_i', \ \dot{\varphi}_i', x^{\mu'})\} d^4x - \int_{R} (\varphi_i, \dot{\varphi}_\iota, x^{\mu}) d^4x. \qquad (3.1)$$

where the integral is over compact space-time regions R and R', bounded by initial and final space –like surfaces, the original of integration R being mapped to the new region of integration R' by the point transformations corresponding to

$$\delta\varphi = \varphi'(x') - \varphi(x) \qquad (3.2)$$

We perform a change of variables in order to make the regions of integration the same:

$$\delta S = \int_{R} \{\mathcal{L}(\varphi_i, \ \dot{\varphi}_\iota, x^{\mu}) + \delta\mathcal{L}(\varphi_i, \ \dot{\varphi}_\iota, x^{\mu})\}\{1 + \partial_\mu(\delta x^{\mu})\, d^4x$$

$$-\int_R \{\mathcal{L}(\varphi_i, \ \dot{\varphi}_\iota, x^{\mu}) d^4x' \ . \qquad (3.3)$$

Where the transformation of the volume element proceeds via the Jacobian $\partial x'/\partial x$ and is correct to first-order. Hence, to first order

$$\delta S = \int_{R'} \{\delta\mathcal{L}(\varphi_i, \ \dot{\varphi}_\iota, x^{\mu}) + \mathcal{L}(\varphi_i, \ \dot{\varphi}_\iota, x^{\mu})\partial_\mu(\delta x^{\mu}) d^4x\}. \qquad (3.4)$$

For the first term, we use Taylor expansions for $\delta\mathcal{L}$ in each of the variables on which $\mathcal{L}$ depends, and again impose the requirement that we are considering first-order variations, therefore restricting ourselves to first-order Taylor expansions. Thus:

$$\delta S = \int \sum_i \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_i} \delta_o \varphi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta_o(\partial_\mu \varphi_i) + (\partial_\mu \mathcal{L})\delta x^\mu + \mathcal{L}\partial_\mu(\delta x^\mu) \right\} d^4x =$$

$$\int \sum_i \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_i} \delta_o \varphi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta_o(\partial_\mu \varphi_i) + \partial_\mu(\mathcal{L}\delta x^\mu) \right\} d^4x. \qquad (3.5)$$

Then, since $\partial_\mu \delta_o \varphi = \partial_\mu \{\varphi'(x) - \varphi(x)\} = \partial_\mu \varphi'(x) - \partial_\mu \delta(x)$

$$= \quad \delta_o\{\partial_\mu \varphi(x)\}.$$

Is $$\delta_o(\partial_\mu \varphi_i) = \partial_\mu(\delta_0 \varphi_i). \qquad (3.6)$$

We get:

$$\delta S = \int \sum_i \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_i} \delta_o \varphi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \partial_\mu(\delta_o \varphi_i) + \partial_\mu(\mathcal{L}\delta x^\mu) \right\} d^4x. \qquad (3.7)$$

$$= \int \sum_i \left\{ \frac{\partial \mathcal{L}}{\partial \varphi_i} \delta_o \varphi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta_o \varphi_i \right) \right.$$

$$\left. - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \right) \delta_o \varphi_i + \partial_\mu(\mathcal{L}\delta x^\mu) \right\} d^4x$$

$$= \int \sum_i \left\{ \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \right) \delta_o \varphi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta_o \varphi_i + \mathcal{L}\delta x^\mu \right) \right\} d^4x$$

Thus, the variation in the action consists of an "interior" term (the first term) and a 'boundary' term (the second terms).

Finally, we reach the following solution of Noether's variational problem:

$$\sum_i \left(\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \right) \delta_o \varphi_i \doteq - \sum_i \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta_o \varphi_i + \mathcal{L}\delta x^\mu \right). \qquad (3.8)$$

Here we have used the symbol '$\doteq$'to indicate that this relation has been derived without using any Euler-Lagrange equations.

If the Euler-Lagrange equations are satisfied by all the dependent variables involved in the variation, then (3.8) becomes:

$$\sum_i \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_i)} \delta_o \varphi_i + \mathcal{L}\delta x^\mu \right) = 0. \qquad (3.9)$$

The analogous expressions for Lagrangians, rather than Lagrangian densities, are:

$$\sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)\delta_o q_i \doteq -\sum_i \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta_o q_i + \mathcal{L}\delta t\right). \qquad (3.10)$$

And, when Euler-Lagrange equations satisfied by all the q_i,

$$\sum_i \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta_o q_i + \mathcal{L}\delta t\right) = 0. \qquad (3.11)$$

This concludes the derivation of the general solution to Noether's variational problem in its original form.

3.1. Euler- Lagrange Equations

Consider independent variables $x = (x_1, \dots, x_n) = (x_\alpha)$ and dependent differential variables $u = (u_1, \dots, u_m) = (u_i)$.

Consider the variation (x, u), compute the variation of $\mathcal{L}$, that of

$$I = \int \dots \int f\left(x, u\ \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \dots\right) dx\ . \qquad (3.12)$$

Use integration by parts to obtain the Euler-Lagrange equation.

In the case $x = t$, $u = q = (q^i)$ we have define the variational derivative of $\mathcal{L}$ also called Euler-Lagrange derivative or Euler-Lagrange differential. Denoted it by $\frac{\delta \mathcal{L}}{\delta q}$ or E$\mathcal{L}$. A necessary condition for a map $t \to q(t)$ with fixed value on the boundary of the domain of integration to minimize the integral (3.12)

Is the Euler-Lagrange equation E$\mathcal{L}$ =0

Lagrangian of order1

$$(\mathrm{E}\mathcal{L})_i = \frac{\delta \mathcal{L}}{\delta q^i} = \frac{d\mathcal{L}}{dq^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\ . \qquad (3.13)$$

The Euler-Lagrange equation in this case is

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = 0\ . \qquad (3.14)$$

Lagrangian of order k

$$(\mathrm{E}\mathcal{L})_i = \frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}^i} - \dots + (-1)^k \frac{d^k}{dt^k}\frac{\partial \mathcal{L}}{\partial q^{i(k)}} = 0. \qquad (3.15)$$

The Euler- Lagrange equation in this case is

$$\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} + \frac{d^2}{dt^2}\frac{\partial \mathcal{L}}{\partial \ddot{q}^i} - \cdots + (-1)^k \frac{d^k}{dt^k}\frac{\partial \mathcal{L}}{\partial \dot{q}^{i(k)}} = 0. \quad (3.16)$$

Notice that is we can put $\frac{\mathrm{d}}{\mathrm{dt^i}} = \mathrm{D_i}$

3.2. Symmetry transformations expressed through invariance of the Lagrangian

We consider a continuous set of time independent coordinate transformations

$$q \to q' = q'(q), \quad (3.17)$$

And assume this to be symmetry transformation in the sense that it leaves the Lagrangian invariant (or unchanged),

$$(q', \dot{q}') = \mathcal{L}(q, \dot{q}). \quad (3.18)$$

This means that the coordinate transformation gives a Lagrangian with same functional dependence of the new and old variables. Since the Lagrangian determines the form of the equations of motion (the Lagrange equation), it follows that a physical time evolution is described by coordinates q(t) and $q'(t)$ that satisfy the same equations of motion . We will demonstrate this explicitly by assuming that if q(t) satisfy Lagrange's equation, then invariance of $\mathcal{L}$ under the coordinate transformation q $\to q'$ implies that q'(t) satisfies the same Lagrange's equations. We consider the change of the variables in the partial derivatives of the Lagrangian

$$\frac{\partial \mathcal{L}}{\partial q'_m} = \sum_k \left(\frac{\partial \mathcal{L}}{\partial q_k}\frac{\partial q_k}{\partial q'_m} + \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q'_m}\right).$$

$$\frac{\partial \mathcal{L}}{\partial \dot{q}'_m} = \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial \dot{q}'_m}. \quad (3.19)$$

Note that in the last expression there is no term proportional to $\partial q_k / \dot{q}'_m$, since in a coordinate transformation the old coordinates q will not depend on the new velocities $\dot{q}'$, but only on the new coordinates q'. The relation between the velocities is

$$\dot{q}_k = \sum_m \frac{\partial q_k}{\partial q'_m}\dot{q}'_m. \quad (3.20)$$

which implies that

$$\frac{\partial q_k}{\partial q'_m} = \frac{\partial \dot{q}_k}{\partial \dot{q}'_m}. \quad (3.21)$$

This allows a reformulation of the partial derivative of $\mathcal{L}$ with respect to velocities

$$\frac{\partial \mathcal{L}}{\partial \dot{q}'_m} = \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k} \frac{\partial q_k}{\partial q'_m} \ . \qquad (3.22)$$

We are interested in the total time derivative

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}'_m}\right) = \sum_k \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right)\frac{\partial q_k}{\partial q'_m} + \sum_k \frac{\partial \mathcal{L}}{\partial \dot{q}_k}\frac{d}{dt}\left(\frac{\partial q_k}{\partial q'_m}\right). \quad (3.23)$$

Where the last term can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial q_k}{\partial q'_m}\right) = \sum_l \frac{\partial^2 q_k}{\partial q'_l q'_m}\dot{q}'_l + \frac{\partial^2 q_k}{\partial t q'_m}$$

$$= \frac{\partial}{\partial q'_m}\left(\sum_l \frac{\partial q_k}{\partial q'_l}\dot{q}'_l + \frac{\partial q_k}{\partial t}\right) = \frac{\partial \dot{q}_k}{\partial q'_m}. \qquad (3.24)$$

We finally collect expressions from (3.19), (3.23), and (3.24), which give the following relation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}'_m}\right) - \frac{\partial \mathcal{L}}{\partial q'_m} = \sum_k \left(\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k}\right)\frac{\partial q_k}{\partial q'_m}. \quad (3.25)$$

This demonstrates explicitly that if $q(t)$ satisfies Lagrange's equations, and thereby the right hand side of (3.25) vanishes, then the transformed coordinates q'(t) will also satisfy (the same set of) Lagrange's equation. Thus, a coordinate transformation that is a symmetry transformation in the sense that it leaves the Lagrangian invariant will also be a symmetry transformation in the sense that it maps solutions of the equations of motion into new solutions. Note, however, that the opposite may not always be true. There may be coordinate transformations that maps solutions of the equations of motion into new solution without leaving the Lagrangian unchanged.

3.3. A Lie group transformations and Noether's theorem

Assume that the Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta u^\alpha} = o \ , \alpha = 1, \dots, m. \qquad (3.26)$$

Admit a one-parameter Lie transformation group G, i.e. a local group of transformations.

$$\bar{\mathrm{x}} = \varphi(\mathrm{x}, \mathrm{u}, \mathrm{a}), \qquad \bar{\mathrm{u}} = \psi(\mathrm{x}, \mathrm{u}, \mathrm{a}). \qquad (3.27)$$

Where $\varphi = (\varphi^1, \dots, \varphi^n)$, $\psi = (\psi^1, \dots, \psi^m)$,

And $\varphi(\mathrm{x}, \mathrm{u}, \mathrm{o}) = \mathrm{x}$, $\psi(x, u, a) = u$

the infinitesimal generator of the group G has the from

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\propto}(x,u)\frac{\partial}{\partial u^{\propto}} \qquad (3.28)$$

where $\xi^i(x,u) = \frac{\partial \varphi^i(x,u,a)}{\partial a}|_{a=0}$, $\eta^{\propto}(x,u) = \frac{\partial \psi^{\alpha}(x,u,a)}{\partial a}|_{a=0}$.

Definition: A variational of integral

$$\frac{\delta \mathcal{L}}{\delta q^{\alpha}} - D_t\left(\frac{\partial \mathcal{L}}{\partial v^{\alpha}}\right) = 0 \; \alpha = 1, \dots, s \qquad (3.29)$$

is invariant under the group G if

$$\int_{\bar{V}} \mathcal{L}\left(\bar{x}, \bar{u}, \bar{u}_{(1)}, \dots, \bar{u}_{(p)} d\bar{x} = \int_{V} \mathcal{L}\big(x, u, u_{(1)}, \dots, u_{(p)}\big)\right) dx \,. \qquad (3.30)$$

The invariance condition is given by the following lemma.

Lemma 3.1 An integral

$$\int_V \mathcal{L}(\mathrm{x}, \mathrm{u}, \mathrm{u}_{(1)}, \dots, u_{(p)}) dx \,. \qquad (3.31)$$

is invariant under the group G if and only if

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = 0 \qquad (3.32)$$

Here X is a prolonged version of the generator (3.28)

$$X = \xi^i \frac{\partial}{\partial_{x^i}} + \eta^{\propto}\frac{\partial}{\partial u^{\propto}} + \varsigma_i^{\propto}\frac{\partial}{\partial u_i^{\propto}} + \cdots + \varsigma_{i_1,\dots,i_s}^{\propto}\frac{\partial}{\partial u_{i_1,i_s}^{\propto}} + \cdots \qquad (3.33)$$

where

$$\varsigma_i^{\propto} = D_i\big(\eta^{\propto} - \xi^i u_j^{\propto}\big) + \xi^i u_{ji}^{\propto} \,,$$

$$\varsigma_{i_1,\dots,i_s}^{\propto} = D_{i_1} \dots D_{i_s}\left(\eta^{\propto} - \xi^i u_j^{\propto}\right) + \xi^i u_{ji_1 \dots i_s}^{\propto} .$$

Noether proved her theorem by the application of the variational procedure to the integral of action. Using her idea Hill presented the explicit form of conserved quantities in the case of the first-order Lagrangians $\mathcal{L}$ (x, u, $u_{(1)}$). We have used the following generalized form of Noether's theorem, proved by Ibragimov on the basis of the group- theoretical approach.

Theorem 3.1 Let the variational integral (3.31) be invariant with respect to a group G with generator (3.28), then a vector C with components

$$T^i = N^i(\mathcal{L}), \ \ i = 1,2,\dots,n \qquad (3.34)$$

is conserved vector for the Euler- Lagrange's equations (3.26) i.e.

$$D_i(T^i)_{(2.12)} = 0 \qquad (3.35)$$

Here N^i are Ibragimov's operator's.

$$N^i = \mathcal{L}\xi^i + W^\alpha\left\{\frac{\partial}{\partial u_i^\alpha} + \sum_{s\geq 1}(-1)^S D_{ji}\dots D_{js}\frac{\partial}{\partial u^\alpha_{ij_1\dots j_s}}\right\} +$$

$$\sum_{\gamma\geq 1} D_{K_1}\dots D_{kr}(W^\alpha)\left\{\frac{\partial}{\partial u^\alpha_{ik_1\dots k_r}} + \sum_{s\geq 1}(-1)^S D_{j1}\dots D_{js}\frac{\partial}{\partial u^\alpha_{ik_1\dots k_{rj1\dots js}}}\right\} \qquad (3.36)$$

Where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

Proof of theorem 3.1. If the variational integral (3.31) i.e.

$$\int_V \mathcal{L}(x, u, u_{(1)},\dots,u_{(p)})dx,$$

is invariant it fulfill $[X(\mathcal{L})]$- $\mathcal{L}D_i(\xi^i)$=0 according to lemma (2.1). By rewriting the operator (2.4) according to equation

$$X = \xi^i D_i + W^\alpha\frac{\partial}{\partial u^\alpha} + D_i(W^\alpha)\frac{\partial}{\partial u_i^\alpha}. \qquad (3.37)$$

Straightforward calculations yield

$$X(\mathcal{L}) + \mathcal{L}D_i(\xi^i) = \xi^i D_i(\mathcal{L}) + W^\alpha\frac{\partial\mathcal{L}}{\partial u^\alpha} + D_i(W^\alpha)\frac{\partial\mathcal{L}}{\partial u_i^\alpha} + \mathcal{L}D_i(\xi^i)$$

$$=\xi^i D_i(\mathcal{L}) + \mathcal{L}D_i(\xi^i) + W^\alpha\frac{\partial\mathcal{L}}{\partial u^\alpha} + D_i\ (W^\alpha)\frac{\partial\mathcal{L}}{\partial u_i^\alpha} - W^\alpha D_i\left(\frac{\partial\mathcal{L}}{\partial u_i^\alpha}\right)$$

$$=D_i(\mathcal{L}\xi^i) + D_i\left(W^\alpha\frac{\partial\mathcal{L}}{\partial u_i^\alpha}\right) + W^\alpha\left(\frac{\partial\mathcal{L}}{\partial u^\alpha} - D_i\frac{\partial\mathcal{L}}{\partial u_i^\alpha}\right)$$

$$=D_i\left(\mathcal{L}\xi^i + W^\alpha\frac{\partial\mathcal{L}}{\partial u_i^\alpha}\right) + W^\alpha\frac{\delta\mathcal{L}}{\delta u^\alpha} \qquad (3.38)$$

We know that for the solutions to the Euler-Lagrange equations we have that $\frac{\delta \mathcal{L}}{\delta u^{\alpha}} = 0$ then, since

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i\left(\mathcal{L}\xi^i + W^{\alpha}\frac{\partial \mathcal{L}}{\partial u_i^{\alpha}}\right) + W^{\alpha}\frac{\delta \mathcal{L}}{\delta u^{\alpha}} = 0.$$

Hence $$D_i\left(\mathcal{L}\xi^i + W^{\alpha}\frac{\partial \mathcal{L}}{\partial u_i^{\alpha}}\right) = 0 \qquad (3.39)$$

According to ($\frac{\delta \mathcal{L}}{\delta u^{\alpha}} = 0$)or $D_i(T^i)\,\Big|_{(3.35)} = 0$

It follows that vector T= $(T^1, \dots, T^n)$ $where\ T^i = (x, u, u_{(1)}, \dots, u_{(p)}$

is conserved vector for $\frac{\delta \mathcal{L}}{\delta u^{\alpha}} = 0$.#

Corollary: If for some one – parameter transformation group the invariance condition (3.32) is not satisfied but the "divergence" condition.

$$X(\mathcal{L}) + \mathcal{L} D_i(\xi^i) = D_i(B^i) \qquad (3.40)$$

Holds, then the components of the corresponding conserved vector, have the form.

$$T^i = N^i(\mathcal{L}) - B^i \ , i = 1, \dots, n \qquad (3.41)$$

Example 4.4. The motion of the free particle in special relativity is described by the relativistic Lagrangian

$$\mathcal{L} = mc^2\sqrt{1-\beta^2} \qquad (4.1)$$

Where $\beta^2 = \frac{|v|^2}{c^2}$, $|v|^2 = \sum_{i=1}^{3}(v^i)^2$. The space coordinates of the particle is

$x = (x^1, x^2, x^3)$ and the velocity is $v = (v^1, v^2, v^3)$ such that v = $\frac{dx}{dt}$, where t is time. We have that x is our differential variables and t is our only independent variable. The generator of the Lorentz group (classical dynamic) is given by

$$X_0 = \frac{\partial}{\partial t}, \quad X_i = \frac{\partial}{\partial x^i}$$
$$X_{ij} = x^j\frac{\partial}{\partial x^i} - x^i\frac{\partial}{\partial x^j}, X_{0i} = t\frac{\partial}{\partial x^i} + \frac{1}{c^2}x^i\frac{\partial}{\partial t}. \qquad (4.2)$$

By applying Noether's theorem to each of these generators with the given Lagrangian we wish to find the corresponding conservation law. We have generators of the form

$$X = \xi(t,x)\frac{\partial}{\partial t} + \eta^{\alpha}(t,x)\frac{\partial}{\partial x^{\alpha}}, \quad \alpha = 1, 2, 3. \qquad (4.3)$$

We will get a conserved quantity of the form

$$T = \xi \boldsymbol{\mathcal{L}} + (\eta^{\alpha} - \xi v^{\alpha}) \frac{\partial \mathcal{L}}{\partial v^{\alpha}}, \quad (4.4)$$

where

$$\frac{\partial \mathcal{L}}{\partial v^{\alpha}} = m v^{\alpha} \gamma \quad , \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad (4.5)$$

for $X_0 = \frac{\partial}{\partial t}$ we have $\xi = 1, \eta^1 = \eta^2 = \eta^3 = 0$. Equation (4.10) yields

$$T = \boldsymbol{\mathcal{L}} - v^{\alpha} \frac{\partial \mathcal{L}}{\partial v^{\alpha}} = -mc^2 \gamma^{-1} - m|v|^2 \gamma = -mc^2 \gamma \,. \quad (4.6)$$

Putting $T = -E$ we get $E = -mc^2 \gamma$, i.e. conservation of energy.

For $X_i = \frac{\partial}{\partial x^i}$ we get a conserved vector $T = T^i = (T^1, T^2, T^3)$ .Here we have $\xi = 0, \ \forall \, i$, while $\eta^{\alpha} = \delta_{i\alpha}$. Equation (4.10) yields

$$T^1 = \frac{\partial \mathcal{L}}{\partial v^1} = m v^1 \gamma$$

$$T^2 = \frac{\partial \mathcal{L}}{\partial v^2} = m v^2 \gamma$$

$$T^3 = \frac{\partial \mathcal{L}}{\partial v^3} = m v^3 \gamma \quad (4.7)$$

Leading to the conserved vector $T = mv\gamma$, i.e. conservation of relativistic momentum.

For the generator $X_{ij} = x^j \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^j}$, we have a rotational symmetry. With $\xi = 0$ and $\eta^{\alpha} = x^j \delta_{i\alpha} - x^i \delta_{\alpha j}$. Equation (4.10) yields

$$T_{12} = x^2 \frac{\partial \mathcal{L}}{\partial v^1} - x^1 \frac{\partial \mathcal{L}}{\partial x^2} = m\gamma(x^2 v^1 - x^1 v^2)$$

$$T_{23} = x^3 \frac{\partial \mathcal{L}}{\partial v^2} - x^2 \frac{\partial \mathcal{L}}{\partial x^3} = m\gamma(x^3 v^2 - x^2 v^3)$$

$$T_{31} = x^1 \frac{\partial \mathcal{L}}{\partial v^3} - x^3 \frac{\partial \mathcal{L}}{\partial x^1} = m\gamma(x^1 v^3 - x^3 v^1). \quad (4.8)$$

We get the conserved vector $T = (T_{23}, T_{31}, T_{12})$. Putting $T = -\boldsymbol{\mathcal{L}}$ we get the conserved vector $\mathcal{L} = m(x \times v\gamma) = x \times \mathrm{p_{rel}}$, i.e. conservation of angular momentum.

For the generator $X_{0i} = t \frac{\partial}{\partial x^i} + \frac{1}{c^2} x^i \frac{\partial}{\partial t}$, we get a vector $T = (T^1, T^2, T^3)$, for the first component we have ξ

$= \frac{1}{c^2}x^1$, $\eta^1 = t$,

$\eta^2 = \eta^3 = 0$. Equation (4. 10) yields

$$T_{01} = \mathcal{L}\frac{1}{c^2}x^1 + (t - \frac{1}{c^2}x^1v^1)\frac{\partial\mathcal{L}}{\partial v^1}$$

$$= \mathcal{L}\frac{1}{c^2}x^1 + mv^1\gamma - \frac{1}{c^2}x^1(v^1)^2\gamma = \frac{m}{\gamma}(-x^1 + tv^1\gamma^2 - x^1[\beta\gamma]^2)$$

$$= m\gamma(tv^1 - x^1). \qquad (4.9)$$

a similar procedure for the remaining two components and putting $T = Q,$ leads to the conserved vector $Q = m\gamma(tx - x),$ i.e. conservation of relative center of mass.

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