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Margin Trading as a Put Option

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Abstract

Margin trading was and still is one of the most important aspect of the retail brokerage business. Moreover, margin trading is the real-life representatives of leverage and short position in financial modeling, which is an important part of academic finance. However, most effort has been on explaining its popularity and assessing empirically its effect on the capital market rather than studying the activities itself. This paper explores a new approach of researching margin by taking advantage of the flexibility and extensive framework of option pricing and shows how it could allow for a deep and robust analysis of margin trading business.

Keywords: Margin trading; put option; brokerage; capital market.

1. Introduction

Margin trading is one of the most popular financial service brokerage firm can provide to its customer and plays a major role in the capital market. The Financial Industry Regulatory Authority records 665 billion of aggregate debit balances by its members at the start of 2018, growing roughly 8% annually since 2014. The need for margin management is especially important in emerging economies. The top securities companies in Vietnam for example, average 25% of revenue from this business alone. As a result of such widespread attractiveness and influence [2], enormous research effort has been made regarding this topic. However, for the most part, the focus has been primarily on solving the puzzle of "why margin trading is popular" as in [12] and its interaction with market efficiency, volatility and volume, notably the recent works of [1;4]. While contributing substantially to our knowledge of the real world, this body of research is still very empirical in nature and therefore has only moderate effect on advancing the theoretical foundation of margin trading. To see the important of margin in financial economics, note that leverage and short position possibility are vital to the assumption of complete market, which in turns creates the foundation for many breakthrough models.

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Most famous example being the capital asset pricing model (CAPM) of [21,10]. And margin itself is our real-world application of leverage/short position, because risk-less and limitless position as in assumed is borderline impossible. Therefore, margin trading is, in author's opinion, essential to the construction of a more inclusive financial model. Another consequence to this lack of attention from the analytical side is that management of margin trading is still very much left for either the broker's self-developed approach or rely on ancient industrial practice that was built without a concrete and systematic groundwork. In contract, the option/derivative market is highly standardized and well developed thanks to the invention of arguably the most important and pivotal model in the financial world, the Black-Scholes option pricing model, and its successors. It quickly became an essential part of the financial world, not only creates channels for firms to hedge and manage their finance, but also provides valuable information for the market. For a brief review on the ingenuity underlining this innovation, see [9]. This paper proposes a theoretical-based solution for said problem by transforming margin trading into a form of option. This early exploration hopefully can aid brokerage firms and financial entities alike to advance their method of managing margin related businesses as well as kick-start the development of incorporating margin trading into the financial market systematically. The paper attempts to accomplish two goals: to develop an early version of a simple, intuitive yet powerful theoretical framework for analyzing margin. And to demonstrate how such approach enables the study of margin trading's characteristics and improves management. The core idea of this paper revolves around leveraging the extensive option pricing literature and applying it into margin trading, effectively treating them as options. This allowed for an endless possibility of employing option techniques into study and analyze margin. The idea is inspired by [16], which utilizes [2] to estimate corporate equity and debt value by treating them like options. Margin equity and debt in each account mirror corporate equity and debt closely and therefore became a viable candidate for such application. This paper aims to adopt Black-Scholes framework purely and faithfully with minimal modification in order to retain the original mode's simplicity and tractability as well as inherit its vast literature and real-world infrastructure. I argue option pricing is the most appropriate framework for margin trading due to two reasons. First, margin trading fit the model better and more naturally than firm value. The equity and debt of each margin account is intrinsically linked to stock prices itself, just like options whereas firm value is unobservable, thus needed to be proxied by market value through certain techniques. Second, the triggering of problematic debt/defaulting events is a huge deal in the corporate world with overwhelming layers of legal battle. This proves to be highly complicating the issue. On the other hand, bad margin debt is highly expected and recovering them is a standardized procedure. Lastly, margin trading duration is closer to option's maturity, around 1 month, as oppose to corporate debt which ranges from a few months to multiple years, avoiding one of biggest weakness of Black-Scholes, it performs worse as time horizon increase. It's important to note that this framework, just like its originator, is built upon theoretical basis and therefore potentially inherits the same empirical limitations. The approach of this paper therefore involves two parts. First and foremost, the mechanical similarity between option and margin trading is established, thereby proving it's theoretically viable to apply option pricing techniques provided certain assumptions and modifications are met. Second, based on the result above, I employ a wide range of option techniques into margin trading to illustrate how it can help solving relevant problems and expand our understanding of how margin trading works. The paper is organized as follow. Section II reviews Merton's paper, layout the assumptions and theoretical framework necessary for developing this model. In section III, those results are applied into margin trading and show how a margin purchase is identical to put option and its implication. The meaning of various term used throughout this paper will also be clarified. Section IV explores potential applications of this model. In section V illustrates how this approach might work by using both real stock prices as well as simulated stock prices. Finally, Section VI discusses possible extensions and further improvement of this approach.

2. Literature review

2.1 The Merton's model

[16] is one of the most pioneering paper in corporate debt valuation. The paper looks at how firm's capital structure resembles that of an option. A simple yet eloquent take on credit spread problem and firm's capital structure problem. The Black-Scholes option pricing model itself requires multiple piece of assumptions. As a result, Merton made the following assumption to enable this method of valuation:

A1. No transactions costs, taxes.

A2. There are a sufficient number of investors.

A3. Borrow at risk-free rate is possible.

A4. Participant can short sale freely.

A5. Interest rate are known and constant overtime.

A6. Trading is continuous in time.

A7. Stock price/Firm value follows a diffusion stochastic process.

In which, assumptions A1 to A4 are related to the "efficient market hypothesis" and can be safely relaxed by alternating the analysis. However, these assumptions are kept anyway for simplicity. Assumption A5 implies no stochastic interest rate which allows for straightforward valuation. Again, Merton noted that modification for different type of interest rate assumption. This leaves assumption A6 and A7 as core assumptions. Continuous trading is related strongly with market microstructure. For example, most markets have close and open time and after-hours trading is generally less liquid, more volatile, and has bigger spread. Follow assumption A7 and assume no dividend payment, stock price process can be described dynamically as:

$$dS = \mu S dt + \sigma S dz \quad (1)$$

In which, S is stock price; μ is the instantaneous expected return on stock price; σ is the instantaneous standard deviation of stock price and dz is a standard Brownian motion process. With the above assumptions, Merton discussion corporate debt valuation under three variation: callable debt, coupon-bearing debt and zero-coupon debt. In this paper's context, our interest is on the valuation of zero-coupon debt due to its simplicity natures and our goal is to lay out the groundwork for our approach. Other result, especially callable debt valuation, can be employed in further development.

Let F be a security whose market value is a function of stock price S and time t, $\mathrm{F} = \mathrm{f}(\mathrm{S}, \mathrm{t})$. By Ito's lemma, security F can be written as:

$$\mathrm{dF} = \left(\frac{\partial F}{\partial x}\mu S + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)\mathrm{dt} + \frac{\partial F}{\partial S}\sigma S dz \quad (2)$$

In which, $\frac{\partial F}{\partial S}$ denotes first order derivative of F with respect to S, $\frac{\partial^2 F}{\partial S^2}$ denotes second order derivative, dz is the same standard Brownian motion process as in (1).

The discrete version of the above equations:

$$\Delta\mathrm{S} = \mu\mathrm{S}\Delta\mathrm{t} + \sigma\mathrm{S}\Delta z \quad (3)$$

$$\Delta\mathrm{F} = \left(\frac{\partial F}{\partial x}\mu S + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)\Delta\mathrm{t} + \frac{\partial F}{\partial S}\sigma S\Delta z \quad (4)$$

where $\Delta\mathrm{t}$ are a small time interval.

Notice that both equations (3) and (4) have the same underlying Brownian process Δz, which mean we can eliminate it by combining security F and stock S. Consider a portfolio including one unit of the security F and a short position of $\frac{\partial F}{\partial S}$ share of the underlying stock. Let Π be the value of this portfolio, the change of Π can be written using (3) and (4):

$$\Delta\Pi = \Delta\mathrm{F} - \partial F/\,\partial S\;\Delta\mathrm{S} \quad (5)$$

$$\Delta\Pi = \left(\frac{\partial\mathrm{F}}{\partial\mathrm{t}} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)\Delta t \quad (6)$$

Since there is no stochastic variable in this equation (the Brownian process is canceled out), this portfolio is riskless and therefore must earn a risk-free return r for the interval of time Δt. It follows that:

$$\Delta\Pi = \mathrm{r}\Pi\Delta t \quad (7)$$

$$\left(\frac{\partial\mathrm{F}}{\partial\mathrm{t}} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2\right)\Delta t = \mathrm{r}\left(F - \frac{\partial\mathrm{F}}{\partial\mathrm{S}}S\right)\Delta t \quad (8)$$

rearrange (8)

$$\frac{\partial\mathrm{F}}{\partial\mathrm{t}} + \frac{\partial\mathrm{F}}{\partial\mathrm{S}}\mathrm{rS} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\sigma^2 S^2 = \mathrm{rF} \quad (9)$$

This is the Black–Scholes–Merton differential equation, which must be satisfied by all securities whose value is a function of stock price and time. In the Black-Scholes option pricing model, this equation is solved using the boundary conditions of option. Merton replicates this result using another set of boundary condition for zero-coupon corporate debt. It turns out Merton's solution is very similar to the Black-Scholes model since the assumed boundary conditions are essentially the same. The assumptions made for corporate debt in Merton's model is as follows: the firm promises to pay B dollars to bondholders at maturity date T; if they can't honor their obligation, bondholders claim ownership of this firm. No coupon payment or issuance of new debt during this period. Let F be the value of debt, V be the value of the firm and apply it into equation (9). In this case, debt is a security whose value depends on its firm value. The boundary conditions can now be derived. If firm value V is zero at expiration, the bond has no value. Therefore, we have the first condition:

$$F(0,T)=0 \tag{10}$$

Second, the value of the bond can't be higher than its pay-off B:

$$\mathrm{F}(V,t)\geq V \tag{11}$$

On maturity date T, the firm have to pay the promised payment B. Otherwise, bondholders earn the right to the firm value V. Therefore, it only makes sense for the firm to refuse paying only if V is less than B at T. Then, the value of debt F is V instead of B. Thus, the last condition is as follow:

$$\begin{aligned}\mathrm{F}(\mathrm{V},0) &= \min[\mathrm{V},\mathrm{B}] = B+\min[V-B,0]\\ &= B-\max[0,B-V]\end{aligned} \tag{12}$$

Using the Black-Scholes differential equation (9) and the three boundary conditions above, one could solve the value of $F(V,t)$ by employing the finite difference method. For more details about the calculations and related discussion of this operation see [2]. Here the established results are applied to solve (9) under (10), (11) and (11) boundary conditions:

$$\mathrm{F}(\mathrm{V},\mathrm{t})=\mathrm{B}e^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}-\left[\mathrm{B}e^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\mathrm{N}(-\mathrm{d}_2)-\mathrm{VN}(-\mathrm{d}_1)\right] \tag{13}$$

$$d_1=\frac{lnV/B+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2=\frac{lnV/B+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Notice that $\left[\mathrm{B}e^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\mathrm{N}(-\mathrm{d}_2)-\mathrm{VN}(-\mathrm{d}_1)\right]$ is the put option pricing formula with stock price V and exercise price B. Using put-call parity, it can be shown that corporate equity value $\mathrm{E}=\mathrm{V}-\mathrm{F}$ have the same solution as the call option pricing formula. In other world, corporate debt and equity can be treated as put and call option respectively. This is the central result of the Merton's model. The mechanism can be simplified as follow: when firm value dips below its debt obligation, the firm can default on this debt and therefore not having to pay it. Defaulting is therefore identical to exercise a put option with pay off equal the difference between firm value and payment duty, then use it to pay up. Equivalently, owning corporate debt is like having a short position on put option and a fixed future cash flow. When the borrower defaults (exercises put), lender loses the difference.

There are obviously unrealistic assumptions made in the original Merton's model. For example, firm's ability to honor debt is determined by its value which follows a diffusion process. This means there will be no jump and thus no surprise default. Nonetheless, the model still offers an eloquent way to interpret corporate liability. A lot of effort and improvement building upon this framework have been made since. One notable contribution is [2]. They replace the default on maturity assumptions which a more accurate safety covenant which allows creditors to take over borrowing firm the moment its value fall below some threshold (as oppose to only at maturity date in Merton's model). As a result, equity is a down-and-out call option instead of a European one. [14] further extends the model to allow for stochastic interest rate. The rich and powerful literature around this topic shows how much potential development such simple approach can be.

2.2 The literature behind option

The most central basis of financial modeling and theory is the efficient market, championed by the work of [7]. The hypothesis states that all relevant information is fully and immediately reflected into price. All investor therefore should not expect to earn an above the risk-adjusted market return through either technical analysis or fundamental analysis. This in turn has powerful implication among both academics and practitioners and create essential premise for financial thinking. The framework employed in this paper is no different. As noted in the previous section, certain assumptions rely heavily on the effectiveness of an efficient market. Despite widespread influence and acceptance, the efficient market hypothesis was constantly under scrutiny and doubt, and for a good reason. Multiple so-called "anomalies" had been documented which gained a lot of tractions and sparked enthusiastic debates among academics. This ultimately gave rise to the development of behavioral finance, led by Nobel-prize winning economist Richard Thalers and his works, notably "A survey of behavioral finance" (2003). These anomalies however aren't free of problem. Reference [15] provides an extensive discussion on how sample specifics, time span, and rational factors can weaken the results of these tests. Reference [11] attempts to "replicate the entire anomalies literature in finance and accounting by compiling a largest-to-date data library that contains 447 anomaly variables" and presents damning evidence against them, suggesting widespread p-hacking problem among the literature. All in all, even though there are evidences against market's perfect efficiency, it's still really close to be. In the words of the famous statistician George E. P. Box, "all models are wrong, but some are useful", a close approximate to reality can be tremendously useful in our advance to understand the world. Volatility is another important topic to the approach introduced in this article. The empirical effect of margin trading on market volatility is well-researched. For example, Reference [19] found that adding stocks into marginable list reduces volatility and noise. In this context however, we are more interested in the fundamental characteristics of them since volatility is the heart and soul of option pricing which in turn connects to margin trading as will later be shown. Fortunately, the central role of option ensures that this topic always receives adequate attention. Let start with a phenomenon generally observed in most security's movement, the heteroscedasticity of stock return (i.e. [19]). The phenomenon implies volatility changes over time and tends to cluster together instead of spreading out evenly. This contradicts the constant volatility assumption and therefore biases standard pricing models. Consequently, a class of times series modeling aimed to incorporate volatility clustering called autoregressive conditional heteroskedasticity (ARCH) models, pioneered by Engle and Robert, exploded in popularity and quickly became standard among practitioners. ARCH, GARCH and other variations can be employed using Monte-Carlo simulation. Next, we have fat-tail distribution, where the extreme movement (commonly known as Black Swan event) is much more likely compare to normal distribution. Therefore, a single standard deviation measures no longer fit to describe price movement. The market cleverly solves such problem by pricing options differently depending on in-the-money/out-of-the-money status. Such occurrence should never exist under if Black-Scholes theory holds true. Reference [10] offers an approach to derive implied distribution from this so-called volatility smile. All in all, volatility has always been and will be a major interest, both academically and practically, in finance and along with them innumerable research, theories and methods which can be of great help in bringing models closer to reality.

2.3 Margin trading as a put option

A. Margin trading definition

The term "margin trading" in this paper will refer to the margin purchase service offered by brokers through a special account called "margin account. In particular, trader can open a margin account and borrow money from their brokers and leverage their stock position. The amount available is depended on a multiple of borrower's cash deposit. This is referred to as margin ratio. For example, with margin ratio 30%, traders need to put upfront $30 to have the purchasing power of $100 (borrow $70). All assets within this account is used as collateral for the debt. The use of margin account is not limited to leveraging stock position. For example, it can be used for shorting, getting into future contact, etc. However, in the context of this paper, I'm more interested in margin purchase as its simplicity makes the best candidate for the early development of this approach. Other usages of margin account can be explored using the same approach developed here in subsequent papers. Margin trading service is risky from the perspective of the firm. If an account value sinks below its debt, broker can take over, sell all holding and lose an amount equal to debt obligation minus sale value. However, margin account often comes with extra safety mechanism. The most popular one is margin call. In event of stock prices within the account fall below certain margin requirement (usually expressed as equity over total asset), trader will receive a margin call from their broker, asking to deposit more cash until the requirement is satisfied. If the trader fails to put up enough money, the broker can forcefully sell off portion of holding. The possibility of cash deposit introduces an extra stochastic variable outside of stock price, and thus a source of risk. Fortunately, the topic of credit modeling is well researched which can be plugged into our model in further development. For now, the focus will be on the risk comes from stock movement.

B. Theoretical framework

First, we echo the assumptions A1-A7 made in Merton's and Black & Scholes model. As mentioned, assumption A1 to A4 can be relaxed in future development. Assumption A5 is fairly innocuous in this context because margin trading term is relatively short. However, it's possible to make modification for stochastic interest rate. Assumptions A6 and A7 are standard in most theoretical finance literature as it allows for easy and straightforward modeling. Since this paper aims at creating a foundation for valuing margin lending business, the mechanics will be simplified as follow. After requesting the service, trader put in cash deposit C which allows them to borrow an amount proportional to margin rate M to buy a specific stock at interest rate R. The borrowed fund would be $C\left(\frac{1}{M}-1\right)$. Trader then use the entire account's value to buy S amount of the stock with $S = C(1/M - 1) - C = C/M$. The holding period T is determined. At maturity, trader makes debt payment of $C\left(\frac{1}{M}-1\right)e^{RT}$, if however, the account's value S is below their obligation, trader will default on this account. Broker then take what leftover, liquidate all stock position and lose the amount $C\left(\frac{1}{M}-1\right)e^{RT} - S$. This assumption about how margin trading works ignore trader's ability to prematurely pay off debt as well as broker's right to demand cash deposit or liquidation of part of trader's stock position. I will come back and discuss this in section IV. Replicate the computation in Section II with S as the price of stock collateralized in the margin account, D as the value of the service from broker's point of view. Because value D depends on underlying stock price S, we have the Black-Scholes differential equation (9) for margin account value:

$$\frac{\partial D}{\partial \mathrm{t}} + \frac{\partial D}{\partial \mathrm{S}}\mathrm{rS} + \frac{1}{2}\frac{\partial^2 D}{\partial S^2}\sigma^2 S^2 = \mathrm{rD} \tag{14}$$

The next step is to derive boundary conditions. First, if the collateralized stock value fall to zero, the account worth nothing to the brokerage firm:

$$D(0,T) = 0 \tag{15}$$

Second, the value of this account can't be higher than its pay-off $C\left(\frac{1}{M}-1\right)e^{RT}$. Let B be the pay-off:

$$D(V,t) \geq \mathrm{B} \tag{16}$$

The value of this account on maturity depends on whether stock value S fall below payment obligation B. If S > B, broker receive full payment of B. Otherwise they receive B minus the difference. The last boundary condition is therefore:

$$D(\mathrm{V},0) = B - \max[0, B - S] \tag{17}$$

Notice how boundary conditions (15), (16), (17) is very similar to the set of conditions (10), (11), (12) in section II. This parallel implies the solution for (14) under these conditions will also be identical to the solution above. Indeed, applying finite difference method, we have:

$$\mathrm{L(V,t)} = \mathrm{Be}^{-\mathrm{r(T-t)}} - \left[\mathrm{Be}^{-\mathrm{r(T-t)}}\mathrm{N}(-\mathrm{d}_2) - \mathrm{SN}(-\mathrm{d}_1)\right] \tag{18}$$

$$d_1 = \frac{lnS/B + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{lnS/B + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

This is the same result as Merton's model. Precisely, the value of a margin trading account from broker's point of view can be interpreted as a fix payment B at maturity, and a short position on put option, with the underlying asset is stock value S and exercise price B. For example, a brokerage firm agrees to let customer A trade on margin $10,000 of stock S at margin ratio 20%, which means they have to deposit $2000 in their margin account. The borrowed amount would be $8000. Risk-free rate is 5%. The broker charge 15% annually and the term is 3 months. The payment customer will have to make at the end is $\$8000e^{0.15*0.25}$ = ~$8300. The present value of this margin account to the firm is therefore equal fixed payment $\$8300e^{-0.05*0.25}$ minus put option with stock price $10000 and exercise price $8300. Let stock S volatility be 30%. This value is:

$$8300e^{-0.05*0.25} - P(10000{,}8300) = 8196 - 57 = 8140$$

The value of this business is $8140 which is higher than the investment of $8000. The firm makes a profit of $8140 by charging 15% interest. Let now consider the same situation, but they only charge 7% interest for the loan. The value is now $7996 which is lower than $8000. This business is unprofitable and therefore should never be done by the brokerage firm. This allowed us to apply the powerful literature of option pricing into valuing and managing all margin related business, as seen in the development and extension of Merton model. Next, the discussion is on some of the most apparent implication of modeling margin trading as put option.

The potential and applications of margin trading as put option:

C. Analyzing interest rate charged

The first topic to be tackled is interest rate charge on margin trading. Currently, cost of borrowing charged by brokers are fixed for all stock position. Here I will examine whether this is efficient pricing or not. Rewrite the formula (18):

$$\mathrm{L(V,t) = Be^{-r(T-t)}N(d_2) + SN(-d_1)} \tag{19}$$

Let R be the interest rate required by lender, we have:

$$L_t = Be^{-R(T-t)} \tag{20}$$

Consequently,

$$R = -\frac{ln(L_t/B)}{T-t} \tag{21}$$

Substitute (19) in (21) minus risk-free interest r, we have the spread R_s. This spread is to compensate for the extra risk broker bares after lending to trader. Note that we’re assuming no defensive mechanism. The actual spread should reasonably expect to be lower.

$$\mathrm{R_s} = -\frac{1}{T-t}\ln\left(\frac{\mathrm{Be^{-r(T-t)}N(d_2) + SN(-d_1)}}{\mathrm{B}}\right) - \mathrm{r}$$

$$\mathrm{R_s} = -\frac{1}{T-t}\ln\left(\mathrm{N(d_2)} + \frac{\mathrm{S}}{\mathrm{Be^{-r(T-t)}}}\mathrm{N(-d_1)}\right) \tag{22}$$

Notice that $\frac{S}{B}$ (value of stock over debt obligation) in equation (22) is related to margin rate M since:

$$\frac{\mathrm{S}}{\mathrm{B}} = \frac{C/\mathrm{M}}{C(1/\mathrm{M}-1)}\mathrm{e^{-R\{T-t)}} = \frac{1}{(1-\mathrm{M})}\mathrm{e^{-R\{T-t)}} \tag{23}$$

Let K be $\frac{S}{B}$, P be $\left(\mathrm{N(d_2)} + \frac{\mathrm{S}}{\mathrm{Be^{-r(T-t)}}}\mathrm{N(-d_1)}\right)$, taking derivative of (22) with respect to K we have:

$$\frac{\partial R_s}{\partial \mathrm{K}} = -\frac{1}{\mathrm{T-t}}\frac{\partial\mathrm{P}/\partial\mathrm{K}}{\mathrm{P}} \tag{24}$$

And,

$$\frac{\partial P}{\partial K} = \frac{1}{\sqrt{2\pi}}e^{-{d_1}^2/2}\frac{S}{B}e^{r(T-t)}\frac{B}{S\sigma\sqrt{T-t}} - \frac{1}{\mathrm{e^{-r(T-t)}}}\mathrm{N(d_1)}$$

$$-\frac{\mathrm{S}}{\mathrm{Be^{-r(T-t)}}}\frac{1}{\sqrt{2\pi}}e^{-{d_1}^2/2}\frac{B}{S\sigma\sqrt{T-t}} + \frac{1}{\mathrm{e^{-r(T-t)}}}$$

$$\frac{\partial P}{\partial K} = -\frac{1}{\mathrm{e^{-r(T-t)}}}\mathrm{N(d_1)} + \frac{1}{\mathrm{e^{-r(T-t)}}} = \frac{1}{\mathrm{e^{-r(T-t)}}}\mathrm{N(-d_1)} \tag{25}$$

Substitute (25) into (24):

$$\frac{\partial R_s}{\partial K} = -\frac{1}{T-t}\frac{N(-d_1)e^{r(T-t)}}{P} < 0 \tag{26}$$

Since $N(-d_1)$, $e^{r(T-t)}$, P are all strictly positive.

Taking derivative of (23) with respect to M:

$$\frac{\partial K}{\partial M} = \frac{1}{(1-M)^2}e^{-R(T-t)} > 0 \tag{27}$$

Since both the denominator and the exponential term are both positive.

We have:

$$\frac{\partial R_s}{\partial M} = \frac{\partial R_s}{\partial K}\frac{\partial K}{\partial M} = -\frac{1}{T-t}\frac{N(-d_1)e^{r(T-t)}}{P}\frac{1}{(1-M)^2}e^{-R(T-t)} < 0 \qquad (28)$$

Inequation (28) shows that, as total margin ratio (cash deposit over purchasing power) increase, interest rate spread (or risk premium) charged by broker decrease. As expected, the more up-front cash is deposited, the safer lending is and thus the lower risk premium. This result is consistent with conventional wisdom. We also can interpret this result using option logic. The margin ratio M dictates exercise price $B = C\left(\frac{1}{M}-1\right)e^{RT}$. The higher M the lower B. And for a put option, lower exercise price means lower value. Remind from previous section that margin trading business is similar to shorting put option. A lower put value is a positive for the firm and thus lower interest rate is required for compensation. Most brokerage firms in the market charge the same interest on any stock, but with different margin ratio (for example, Table 1 shows margin ratio for different stocks by Saigon Securities). This mind seems puzzling as first, because we would expect them to charge lower interest for higher margin. However, there is another factor in play that we've yet to analyze, stock price volatility.

Table 1: Saigon Securities Margin Ratio.

Ticker	Company Name	Exchange	Ratio
AAA	An Phat Plastic & Green Environment	HSX	0.3
BVH	Bao Viet Holdings	HSX	0.3
CSM	The Southern Rubber Industry	HSX	0.2
DBC	Dabaco Group	HSX	0.4
GMD	Gemadept Corporation	HSX	0.5
PTB	Phu Tai Joint Stock Company	HSX	0.5
SFI	Sea & Air Freight International	HSX	0.1
VGC	Viglacera Corporation	HNX	0.4
TNG	TNG Investment and Trading	HNX	0.3
SHB	Saigon Hanoi Commercial	HSX	0.5
VIC	Vingroup	HSX	0.3
VNM	Viet Nam Dairy Products	HSX	0.5

Taking derivative of (22) with respect to σ we have:

$$\frac{\partial R_s}{\partial \sigma} = \frac{1}{T-t}\frac{\mathrm{SN}'(\mathrm{d_1})\sqrt{(\mathrm{T-t})}}{P} > 0 \qquad (29)$$

Since $\mathrm{N}'(d_1)$, $\sqrt{(T-t)}$, P are all strictly positive.

Inequation (29) shows that, as stock price volatility σ increase, risk premium charged by broker increase. This relationship is a direct result of treating margin trading business as a short position on put option. Using similar logic as above, higher volatility -> higher put price -> lower value for the firm -> require higher interest. Now that we conclude volatility affects risk premium positively, the puzzle mentioned previously can be easily explained as follow: in order to keep interest rate fixed among different stocks (which mind be desirable because fixed rate is more convenient for customers), the firm can simply require higher margin ratio for high volatility stock and vice versa. The opposite impact these two factors have cancel out each other, which leaves risk premium unchanged. A natural consequence is we can estimate brokerage firm's relative volatility expectation for certain stocks by analyzing their margin ratio and interest rate. From broker's perspective, they can employ my model to estimate what level of margin ratio is appropriate for each individual stock.

D. Delta hedging

Hedging is a major part of risk management in many financial businesses. Moreover, hedging is sometime legally required by regulator. For example, in Vietnam, issuer of cover warrant product is mandated by the State Security Commission to hedge against any stock price exposure incurred. As mentioned, for a margin trading account, the broker has two type of risk: the risk associate with stock price movement and the specific customer risk. With my approach, the former can be hedged using a similar hedging scheme to option. This is especially important to situation where a firm conducts business in both margin trading and selling covered warrant of the same stock. Exposure from margin trading must not be ignored to prevent over-hedging. Remember from section II that we can create an instantaneous risk-less portfolio by going long short in two different assets with the same underlying exposure, so they cancel out each other. In this case, we have margin trading as shorting put and therefore can be hedged using the underlying stock. First, calculate the delta of a margin trading account by taking derivative of (19) with respect to stock price S:

$$\frac{\partial L}{\partial S} = N(-d_1) \tag{30}$$

which is the same as a short put delta. Equation (30) implies that when the value of collateralized stock S increase by ΔS, the value of the margin trading account will also increase by $\mathrm{N}(-d_1)\Delta S$. Therefore, in order to hedge out exposure to the underlying stock, broker need to take a position on stock S equal to: $-N(-d_1)$ with $d_1 = \frac{lnS/B+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$. Hedging scheme can be expanded to gamma, vega, theta hedging just as we already do with option. Once again, brokerage firm can now take advantage of the rich literature behind option hedging and apply it into managing their business.

E. Monte-Carlo simulation

One of the most powerful technique that option pricing (or any risk neutral valuation) makes available is the use of Monte-Carlo simulation. The tool allows extremely detailed and flexible mechanics into our model. But the most attractive feature in my opinion is how easy it is to use. Running a Monte-Carlo simulation doesn't demand strong analytical or mathematical knowledge. As long as user understands the mechanics behind the security they want to value, the whole process should be done relatively straightforward. Now that we have established a part of margin trading account can be valued using risk neutral pricing approach similar to option, there is little reason not to take advantage of this useful tool. We will show how additional protections of the margin lending activity can be dealt with in a simple and intuitive way. But first, we start with a quick reminder of how Monte-Carlo simulation works for valuing option, for more discussion on using applying the simulation for option, see [3]: Step 1, we simulate the stock price process starting with the input value S as S_0. There plethora of model available to choose from, from random-walk to jump diffusion stochastic process. Here we stick with the assumption (1):

$$\mathrm{dS} = \mu S dt + \sigma S dz$$

Specifically, we use equation (1) to calculate S_1 from S_0, S_2 from S_1 and so on until we reach S_T with T as the maturity of the option. Step 2, Calculate the payoff at time T. The payoff is equal $max(S_T - K, 0)$ with K as the exercise price of the option. Step 3, Repeat Step 1 and 2 thousand of times then calculate the average value of payoff.

1. Force selling

Section III mentioned two popular safety mechanism, one of them is the ability to force a sale of trader current position. In particular, this is the right of the brokerage firm to sell their customer's stock without notice once the margin ratio drops to certain level called margin requirement. This acts as a protection from risk because it reduces their exposure to stock downward movement, while simultaneously increase the threshold where stock price must fall into in order to make the account goes under. Using option interpretation of margin trading, we can see this is the same as the right to exchange their current short position on put option into another one with lower value (positive for the firm). To see it clearly, consider the same situation as in the example at the end of section II, where the margin ratio is 20%, stock value is \$10000, debt payment \$8300. Now the firm requires the trader to sell their position whenever margin ratio dip below 12% until it returns to 20%. Say after a few weeks, the stock value and cash deposit (equity) fall by \$1000 to \$9000 and \$1000 respectively. the ratio is then \$1000/\$9000 = ~11% which is lower than 12%. The call is made, and trader sells \$666 of his stock for cash deposit in order to get the ratio back to 20%. His stock position is now \$8333 but his cash deposit increases to \$1666. His debt obligation therefore downs twice the sale amount, to \$6967, since not only trader reduces their position, they have to put the cash from sale into collateral as well. See Table.2 for illustration.

Table 2: Force selling event.

Table.2 Force selling event.					
	Stock value	Borrowed fund	Actual Debt	Cash	Equity
t=1	\$10,000	\$8,000	\$8,300	\$2,000	\$1,700
t=15	\$9,000	\$8,000	\$8,300	\$1,000	\$700
Stock sold	\$666.00				
t=15	\$8,333	\$6,667	\$6,967	\$1,666	\$1,366

In this case, the broker short position on put with S = \$9000, B = \$8,300 turns in to another short position on put with S = \$8300, B = \$6,967.

The formulas for S and B are:

$$S = \frac{2S_{before} - (1-M)S_0}{1+M} \quad (31)$$

$$B = B_{before} - 2\frac{MS_{before} - \left(S_{before} - (1-M)S_0\right)}{1+M} \quad (32)$$

With B_{before} and S_{before} are the value of S and B before being modified by the sale. One consideration around this ability is a perfectly executed force selling scheme would in theory eliminate risk completely. In order for the lender (brokerage firm) to lose money, the account's equity must fall below zero, otherwise they can simply sell all asset and fully recover payment obligation. However, the process in reality is usually much more complicated. Problems such as liquidity, market crash, trader's resistance, etc. prevent this protection from working perfectly. Therefore, firm should assess their ability to execute realistically and alter the mechanics appropriately. Now that I've shown how to make adjustment for force selling mechanics, It can be applied into the Monte-Carlo simulation, by simply adding an "if-then" statement. If total stock value S is lower than some threshold determined by the broker's requirement, it automatically adjust according to (31) and (32) for the rest of the process (or until another event like this happens). By comparing with and without force selling mechanics, we can examine the value of this protection.

2. Additional cash deposit

The other form of protection mentioned previous is additional cash deposit. If traders don't want to see their position reduced, they can agree to put more money into their account equity, thereby restoring the margin ratio to required value. And when that happens, similarly to the case above, we can simply adjust the characteristics of the short put position. In fact, it's a lot simpler. All that's needed is to decrease the debt obligation B by the cash needed to bring margin ratio back to requirement level. Continue with the previous example, now instead of letting broker forces the sale, trader agree to put up more cash deposit. The amount needed is \$800. Consequentially, our put option's B value changes to \$7500, the total stock position S remains the same. See Table 3.

Table 3: Cash payment event.

Table.3 Cash payment event.					
	Stock value	Borrowed fund	Actual Debt	Cash	Equity
t=1	\$10,000	\$8,000	\$8,300	\$2,000	\$1,700
t=15	\$9,000	\$8,000	\$8,300	\$1,000	\$700
Cash paid				\$8,00	
t=15	\$9,000	\$7,200	\$7,500	\$1,800	\$1,500

The formula for this adjustment is:

$$B = B_{before} - \left((M-1)S_{before} + S_0 M\right) \quad (33)$$

Assuming the probability of payment is 40%, the value of the margin trading account to broker can be estimated using Monte-Carlo simulation the same way as above, by adding an "if-then" trigger with 40% chance of triggering. The probability assumption made is the simplest one. Fortunately, there exists a variety of sophisticated methods of modeling available from the credit management literature. Applying them into the simulation should be straightforward as well. Now that we know how to individually deal with each situation, both can easily be combined. 60% the first event applies, 40% the second. See next section for detailed illustration. It's important to note that results obtained from using Monte-Carlo simulation for cash payment possibility is only valid in a risk-neutral world, as it does not take into account the risk premium from un-hedgeable random event. Therefore, firms should only use it to assess the extra risk they would have taken by including an exposure to specific traders/customers. Monte-Carlo can also be used to measure Greek letters of a margin trading account including all aforementioned protections. Remind that Delta of an option measures the degree to which an option is exposed to shifts in the price of the underlying stock. Our first guess usually would be to simply alter stock price S by a small amount and recalculate the whole thing. However, this leaves our estimation vulnerable due to the random nature of Monte-Carlo simulation. We can solve this problem by simply keeping all the stochastic variable generated from the previous calculation (in this case, the Brownian process *dz*) and redo the simulation with this same set of variables. As a result, we can measure how the value of margin trading account would change as stock position slightly change under the exact same condition.

3. Illustrations

F. The value of force-selling and cash deposit as protection mechanics

This section starts with an illustration on how force selling can benefit the brokerage firm. As discussed above, force selling should in theory improve the value of a margin trading account by reducing the value of the put option for the trader. Here I apply the calculation described in the previous section to various stocks currently available for margin trading in Vietnam, using real margin ratio as required by brokerage firms. The amount requested by client is \$10,000 for 3 months. The volatility of this stock is calculated from one-year daily return, interest rate charged for all borrowing is 15%. Margin call happens when margin ratio dip 5 percentage point below requirement. I choose a bundle of different stocks with different margin requirement and volatility. The results presented in Table. 4. t indicates how many times the mechanics will be allowed to take effect, t=0 means no protection. Column (4) in Table. 4 calculates the present value of fixed cash payment portion of the margin trading account. In other word, the profit firms would earn if their margin lending business has no risk. Column (5), (6) and (7) shows how powerful this protection can be. The value of the put option held by trader falls significantly even if force selling only allowed to trigger once (t=1). And continue to fall further as it is allowed to happen twice. This translates to an increase in profitability of the business as shown in column (8), (9) and (10). It appears that, in term of profit, protection is much more valuable in case of higher leverage (low margin ratio). The effects on account with low leverage is negligible. This is expected since the safer an account is, the less likely this protection is ever needed.

Table 4: Valuing margin trading account with force selling protection.

Table. 4. Valuing margin trading account with force selling protection.									
Ticker	Margin ratio	Volatility	PV fixed only	Put value			PV with put (4)-(5)(6)(7)		
(1)	(2)	(3)	(4)	(5) t=2	(6) t=1	(7) t=0	(8) t=2	(9) t=1	(10) t=0
NVB	0.2	0.36	200	16	43.3	107	183.7	156.4	92.7
BCE	0.2	0.29	200	3.8	14.7	49	195.9	185.0	150.7
HMH	0.2	0.24	200	0.86	5	20.7	198.9	194.7	179.0
AAA	0.3	0.45	175	7.5	22.3	56.9	167.2	152.4	117.8
BCG	0.3	0.35	175	1	4	14.9	173.7	170.7	159.8
DVP	0.3	0.26	175	0	0.2	1.6	174.7	174.5	173.1
HHS	0.4	0.42	150	0.21	1.17	4.9	149.6	148.6	144.9
AST	0.4	0.36	150	0.04	0.18	1.5	149.7	149.6	148.3
BVS	0.4	0.28	150	0	0	0.09	149.8	149.8	149.7
SII	0.5	0.66	125	1.9	6.2	17.7	122.9	118.6	107.1
DIG	0.5	0.49	125	0	0.3	1.4	124.8	124.5	123.4
PAN	0.5	0.35	125	0	0	0	124.8	124.8	124.8

Table. 5 presents the same result, but with an addition of cash deposit mechanics. Specifically, every time margin call is made, trader has a 40% chance of adding more cash. If not, the broker proceeds to sell off part of their stock position. Surprisingly, adding cash deposit possibility does not improve the profitability of the margin trading account for broker. In fact, some put value seems to be even higher. This result implies that cash deposit as a protection mechanics isn't inherently superior to force selling and therefore sharing combining the two mechanics (at 40% and 60%) does not lead to better protection. Moreover, some interest observations are revealed from this result. For example, the value of the put deteriorated quickly (good for the firm) as margin ratio rises while interest earned also decrease (bad for the firm). The two contradicting forces means requiring more cash up front might not always be value-adding, firms should do careful analysis before deciding the terms of margin accounts. A more throughout analysis of margin trading value is out of the scope of this paper. However, as seen in this result, it could uncover valuable information to aid firms in managing their business.

Table 5: Valuing margin trading account with 40% cash deposit and 60% force selling protection.

Table. 5. Valuing margin trading account with 40% cash deposit and 60% force selling protection.									
Ticker	Margin ratio	Volatility	PV fixed only	Put value			PV with put (4)-(5)(6)(7)		
(1)	(2)	(3)	(4)	(5) t=2	(6) t=1	(7) t=0	(8) t=2	(9) t=1	(10) t=0
NVB	0.2	0.36	200	17	43.3	107	182.7	156.4	92.7
BCE	0.2	0.29	200	4.2	15.3	49	195.5	184.4	150.7
HMH	0.2	0.24	200	0.91	4.8	20.7	198.8	194.9	179.0
AAA	0.3	0.45	175	7.47	21.9	56.9	167.3	152.8	117.8
BCG	0.3	0.35	175	1.02	4.2	14.9	173.7	170.5	159.8
DVP	0.3	0.26	175	0	0.23	1.6	174.7	174.5	173.1
HHS	0.4	0.42	150	0.22	1.2	4.9	149.6	148.6	144.9
AST	0.4	0.36	150	0.05	0.21	1.5	149.7	149.6	148.3
BVS	0.4	0.28	150	0	0	0.09	149.8	149.8	149.7
SII	0.5	0.66	125	1	5.8	17.7	123.8	119.0	107.1
DIG	0.5	0.49	125	0	0.24	1.4	124.8	124.6	123.4
PAN	0.5	0.35	125	0	0	0	124.8	124.8	124.8

G. Hedging

This section examines how a hedging scheme based on our approach can help brokerage firm protect themselves from losses. To do so, we simulate a situation where brokers actively hedge against the put options in their margin trading accounts using delta calculated by the method described in section IV then compare their loss to the base case where no hedging involved. We obtain stock price data for several stocks out of a pool of 180 stock available for margin trading with emphasis on whether they experienced major drawdown during the last three months. The reason for this selection is to highlight cases where brokers would suffer loss and how a hedging scheme might prevent it. The details for each step in the hedging scheme are as follow:

Step 1, trader engage in margin trading business with the firm. Buying $10,000 worth of stock with 3 months maturity.

Step 2, the firm estimates delta of this margin trading account based on current stock value and takes hedge position accordingly.

Step 3, after one day, calculates gain/loss from hedge position then repeat Step 2 for today stock price.

Step 4 repeat Step 3 until maturity.

Step 5 calculate total gain/loss.

Assuming there is only one form of protection, force selling which can trigger once (t=1) and twice (t=2). The justifications for this choice are first, having cash payment doesn't seem to affect price as much and second, we avoid the nuance of having an extra source of stochastic variable. Table. 6 shows how our hedging scheme performs under real stock price innovation for the past 3 months (13/08/2018 - 12/11/2018).

Table 6: Hedge performance.

Table. 6. Hedge performance.

Ticker		Loss from short put position	Gain/Loss from hedge position	Total Gain/Loss (3)+(4)	% Stock price change
(1)	(2)	(3)	(4)	(5)	(6)
THT	t=1	0	-47	-47	12%
	t=2	0	-19	-19	
AAA	t=1	0	70	70	-13%
	t=2	0	35	35	
TA9	t=1	0	312	312	-25%
	t=2	0	68	68	
TDG	t=1	-659	323	-336	-39%
	t=2	-10	194	184	
HCD	t=1	-1,948	673	-1,274	-43%
	t=2	-1,101	407	-694	
HHG	t=1	-1,721	719	-1,002	-43%
	t=2	-1,178	455	-723	
SJF	t=1	-2,768	1,160	-1,608	-54%
	t=2	-1,864	771	-1,093	
NSH	t=1	-3,476	1,199	-2,277	-62%
	t=2	-2,777	780	-1,997	

Column (6) shows the % change in stock price after 3 months period to give an idea how it translates to the loss in column (3). We can see that a hedging scheme does indeed reduce the loss that broker otherwise would take. However, the loss prevention isn't perfect as it usually only depresses loss by around 40%. This observation can be explained by the lack of gamma hedging employed. Delta hedging only performs well with small movement of price. However, as seen in column (6), stock price needs to move significantly to eat away all equity of the account.

4. On future development

This section discusses some of the limits and potential improvement of the model for future research. One of the most glaring differences between the assumptions we have made so far and how margin trading work is that trader can default on their debt before maturity. Fortunately, this problem isn't new in the literature of credit modeling. For example, as in [8], if we know beforehand the level in which trader will default, the put option in our margin trading account will instead become a down-and-out put option. Trader are also allowed to close out position and pay off debt early. This is equivalent to abandoning their put option, but at the same time, reducing their interest payment. The former is positive to the firm, since they hold the short position. The later however is negative, because of the interest earning loss. We can explore this idea in subsequent papers. Another important aspect of the business is that margin trading usually involves more than one stocks. This might have strong implication to valuation because of correlation between different stocks and diversification effect. One potential solution is to calculate the first and second moment of the distribution of the basket of stocks as explained in [10]. In order to make that adjustment, we must have a correlation structure between relevant stocks. Many solutions are available, from a simple multivariable normal distribution, to Gaussian Copula Model. Firms using self-developed risk model can plug in their own correlation matrix as well. Another approach is to simulate the value of this basket, bases on pre-determined correlation structure and use them for Monte-Carlo valuation. This approach is not limited to margin trading as it can be expanded into other part of the brokerage services like shorting, future/option contract, etc. All are conducted under the same margin account. For example, a short position in margin account is equivalent to a long one but in the opposite direction. In order world, it would be similar to a fixed payment amount plus a short position on a Call option from the broker's point of view. For future contract, a good guess would be a combination of put and call options with different strike price. One of the most powerful insight this method provides is the ability to analytically separate different forms of risk from a single package. As discussed above, the risk come from two sources, stock price movement and trader specific. Once one source of risk is established, it opens up the ability to extrapolate into the other. Brokerage firm can then carefully and thoroughly study their exposure to find improvement and develop appropriate management strategy. Therefore, we believe much research effort is worthwhile in this aspect.

5. Conclusion

The paper proposes a theoretical framework for treating margin trading business as a put option, building on top of [16]. Simple and intuitive, it approaches the problem by building a strong analytical foundation rather than starting with empirical results. This makes practical sense since margin data is not publicly available. The article then explored some potential use of this new approach as well as illustrated how it can reveal valuable information. The valuation analysis confirms many conventional beliefs about margin. Specifically, a higher margin loan ratio is beneficial to the broker whereas increase volatility is detrimental (and vice versa for margin trader). This isn't exactly valuable as most firms are aware. However, now that these relationships can be analytically determined, they can comfortably decide the appropriate level of interest for each stock which opens the possibility of dynamic pricing. Customers then might freely choose their leverage instead of being limited to broker's offer. On the other hand, firms who wish to keep their offers simple to avoid overwhelming clients have the tool to adjust other factors and ensure they are not getting bad deals. The value of defensive mechanism, force-selling and cash deposit, are briefly examined using the new framework. This value can range from basically makes or breaks profitability to doesn't matter at all, depending on other factors. Interestingly, the analysis shows that asking traders to deposit cash before force-selling doesn't necessarily improve value in a significant way. Therefore, cash deposit only serves as a mechanism to allow traders to maintain their position. Whether they refuse to pay up or not doesn't seem to cause the broker losses. This article also shows how hedging can help firms manage their positions. Due to the wide safety margin most brokerage firms prefer when conducting this business, margin trading is mostly deep out-of-the-money put option. For them to get into exercise range, a big stock price movement is required and therefore, second-order approximation (gamma) should be adequately employed. The ability to hedge also means firms don't have to force-sell every time the threshold is hit and leave traders a bit of a breathing room. This by itself does nothing to the value of the margin business, however it may help improve client's experience and satisfaction. Finally, we discussed opportunities for further research and extension of the model. we believe this development will greatly improve brokerage firm’s ability to manage, control and conduct margin related business.

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