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Free Vibration Analysis of Isotropic Plates by Alternative Hierarchical Finite Element Method Based on Reddy's C1 HSDT

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Abstract

This paper presents the free vibration analysis of isotropic thick rectangular plates, based on higher order shear deformation theory (HSDT). The plate theory ensures a zero shear-stress condition at the top and bottom surfaces of the plate, and do not requires a shear correction factor. The model requires inter-element C1 continuity for the transverse displacement. To overcome this hindrance, a new hierarchical p-element with six degrees of freedom per node is developed and used to find natural frequencies of thick plates. Convergence studies and comparison have been carried out for with different boundaries conditions. It is shown that the present element enables rapid convergence.

Keywords: Free vibration; Thick isotropic plates; hierarchical finite element method; third order C1 HSDT.

1. Introduction

Thick plates are extensively used in many fields of engineering, including aerospace, civil structures, hydraulic structures, etc. For plates analysis different theories exists, the classical plates theory (CPT) is adopted for thin plates, where the effect of shear deformation is neglected [1]. The Reissner. Mindlin plate theory is used for moderately thick plates, known as the first order shear deformation theory (FSDT), in which the effect of shear deformation is considered by using a proper choice of a shear correction factor which depend on loading and boundary conditions [2]. The simplifying assumptions made in CPT and FSDT are reflected by the high percentage errors in the results of thick plates analysis. For these plates, higher-order shear deformation theories (HSDT) are required. The HSDT ensure a zero shear-stress condition on the top and bottom surfaces of the plate, and do not require a shear correction factor, which is a major feature of these theories.

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Nelson and Lorch [3], the authors in [4] presented a HSDT for laminated plates however the displacement field does satisfy the shear-stress free condition on the top and bottom surfaces of the plate. Lewinson [5], Murthy [6], and Reddy [7] presented a new higher order shear deformation theories considered as an extension of hencky's theory, which include a realistic displacement field satisfying the conditions of zero transverse shear-stress and/or strains, known as Reddy's third-order theory. This model requires C1 inter element continuity requirement. Phan and Reddy developed a non conforming rectangular element with seven degrees of freedom per node, based on C1 Reddy's third order theory to analyze laminated composites plates. Kant and his colleagues [8] investigate the free and transient vibration analysis of composites and sandwich plates based on a refined theory by using the finite element method and analytical solution. The authors in [9,10] investigate the free vibration and transient response of composite sandwich plates by using two C0 assumed strain finite element based on Reddy's third-order theory. Sheikh and Chakrabarti [11] used a triangular element based on Reddy's higher order shear deformation plate theory. Batra and his colleagues . [12] used a HSDT and the finite element method to analyze free vibrations and stress distribution in tick isotropic plate. Kulkarni and Kapuria [13] used a discrete Kirchoff quadrilateral element based on the third order theory for composite plates.

Because Reddy's third-order theory requires inter element C1 continuity on the transverse displacement. The conclusion can be made from the literature review, that a very few conforming element based on this plate theory are developed. To overcome this hindrance, the hierarchical finite element method can be used. In the hierarchical finite element method the mesh keeps unchanged and the polynomial degree of the shape functions is increased. See for instance Szabo and Sahrmann [14], Szabo and Babuska [15] and Hamza-Cherif [16].

In this paper we address these above-mentioned points. The new approach with hierarchical finite element method is formulated for thick plates vibration analysis. A new hierarchical p-element with six degrees of freedom per node is developed, based on the C1 higher order shear deformation theory. The continuity along the inter-element boundary is not required in the model. To demonstrate the convergence and accuracy of the proposed method, present results are compared with existing data available from other analytical and numerical methods. Then, natural frequencies of rectangular plates under different boundary conditions are tabulated for a wide range of aspect ratios and thickness to length ratios.

2. Formulation

1.1. Energy formulation

Consider an homogeneous, isotropic, thick plate bounded by $0 \leq x \leq a$, $0 \leq y \leq b$, and $-h/2 \leq z \leq h/2$, as show in Fig. 1.

The displacement of the plate are decomposed into three orthogonal components, u, v and w parallel to the x-axis ,y-axis and z-axis, respectively.

In accordance with the higher-order shear deformable theory [10,11], the displacements can be expressed as

$$u = z\,\theta_x - f(z)\left(\frac{\partial w_0}{\partial x} + \theta_x\right)$$

$$v = z\,\theta_y - f(z)\left(\frac{\partial w_0}{\partial y} + \theta_y\right) \quad (1)$$

$$w = w_0$$

In which $$f(z) = \frac{4z^3}{3h^2} \quad (2)$$

Where w_0 is the transverse displacement of middle plate components and θ_x, θ_y are the rotations of the normal to the middle plane about the x-axis and y-axis respectively.

The linear strain-displacement relationships.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial w_0}{\partial y} + \theta_y \\ \frac{\partial w_0}{\partial x} + \theta_x \\ 0 \end{Bmatrix} + z \begin{Bmatrix} \frac{\partial \theta_x}{\partial x} \\ \frac{\partial \theta_y}{\partial y} \\ 0 \\ 0 \\ \frac{\partial \theta_y}{\partial x} + \frac{\partial \theta_x}{\partial y} \end{Bmatrix} - \frac{\partial f(z)}{\partial z} \begin{Bmatrix} 0 \\ 0 \\ \frac{\partial w_0}{\partial y} + \theta_y \\ \frac{\partial w_0}{\partial x} + \theta_x \\ 0 \end{Bmatrix} - f(z) \begin{Bmatrix} \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial \theta_x}{\partial x} \\ \frac{\partial^2 w_0}{\partial y^2} + \frac{\partial \theta_y}{\partial y} \\ 0 \\ 0 \\ \frac{\partial \theta_y}{\partial x} + \frac{\partial \theta_x}{\partial y} + 2\frac{\partial^2 w_0}{\partial xy} \end{Bmatrix} \quad (3)$$

The constitutive equations for linear elastic isotropic material are

$$\{\sigma\} = [C]\{\varepsilon\} \quad (4)$$

In the case of plane stress the stress vector can be written as

$$\{\sigma\} = \{\sigma_{xx} \quad \sigma_{xx} \quad \tau_{yz} \quad \tau_{xz} \quad \tau_{xy}\} \quad (5)$$

Where

$$[C] = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 & 0 & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} & 0 & 0 \\ 0 & 0 & 0 & \frac{E}{2(1+\nu)} & 0 \\ 0 & 0 & 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} \quad (6)$$

Here $C_{i,j}$ are the material property coefficients, in which E, and v are the Young's modulus and, Poisson's ratio, respectively.

The kinetic energy of a bending vibrating thick plate is given by

$$Ec = \frac{1}{2} a\, b \int_0^1 \int_0^1 \left[\frac{1}{a^2} \rho_D \left(\frac{\partial \dot{w}_0}{\partial \xi} \right)^2 + \frac{2}{a} \rho_D \frac{\partial \dot{w}_0}{\partial \xi} \dot{\theta}_x + \rho_F \, \dot{\theta}_x^{\,2} + \frac{1}{a^2} \rho_D \left(\frac{\partial \dot{w}_0}{\partial \eta} \right)^2 + \frac{2}{a} \rho_D \frac{\partial \dot{w}_0}{\partial \eta} \dot{\theta}_y + \rho_F \, \dot{\theta}_y^{\,2} + \rho_A \dot{w}_0^{\,2} \right] d\xi \, d\eta \tag{7}$$

Where ρ is the mass density per unit volume.

$$\rho_A = \int_{-h/2}^{h/2} \rho \, dz, \qquad \rho_D = \int_{-h/2}^{h/2} \rho \, f(z)^2 \, dz, \qquad \rho_F = \int_{-h/2}^{h/2} \rho \left(f(z)^2 - 2\, z\, f(z) + z^2 \right) dz \tag{8, 9, 10}$$

The strain energy of a thick plate is expressed as

$$\begin{aligned} Ed = \frac{1}{2} a\, b \int_0^1 \int_0^1 \Bigg[& \frac{D_{11}}{a^4} \left(\frac{\partial^2 w_0}{\partial \xi^2} \right)^2 + \frac{D_{11}}{b^4} \left(\frac{\partial^2 w_0}{\partial \eta^2} \right)^2 + \frac{2 E_{11}}{a^3} \left(\frac{\partial^2 w_0}{\partial \xi^2} \frac{\partial \theta_x}{\partial \xi} \right) + \frac{2 E_{11}}{a^3} \left(\frac{\partial^2 w_0}{\partial \eta^2} \frac{\partial \theta_y}{\partial \eta} \right) \\ & + \frac{F_{11}}{a^2} \left(\frac{\partial \theta_x}{\partial \xi} \right)^2 + \frac{F_{11}}{b^2} \left(\frac{\partial \theta_y}{\partial \eta} \right)^2 + \frac{2 D_{12}}{a^2 b^2} \left(\frac{\partial^2 w_0}{\partial \xi^2} \right) \left(\frac{\partial^2 w_0}{\partial \eta^2} \right) + \frac{2 E_{12}}{a\, b^2} \left(\frac{\partial^2 w_0}{\partial \eta^2} \frac{\partial \theta_x}{\partial \xi} \right) \\ & + \frac{2 E_{12}}{a^2\, b} \left(\frac{\partial^2 w_0}{\partial \xi^2} \frac{\partial \theta_y}{\partial \eta} \right) + 2 F_{12} \left(\frac{\partial \theta_x}{\partial \xi} \frac{\partial \theta_y}{\partial \eta} \right) + \frac{D_{12}}{a^4} \left(\frac{\partial^2 w_0}{\partial \xi^2} \right)^2 + \frac{D_{12}}{b^4} \left(\frac{\partial^2 w_0}{\partial \eta^2} \right)^2 \\ & + \frac{2 E_{12}}{a^3} \left(\frac{\partial^2 w_0}{\partial \xi^2} \frac{\partial \theta_x}{\partial \xi} \right) + \frac{2 E_{12}}{b^3} \left(\frac{\partial^2 w_0}{\partial \eta^2} \frac{\partial \theta_y}{\partial \eta} \right) + \frac{F_{12}}{b^2} \left(\frac{\partial \theta_x}{\partial \xi} \right)^2 + \frac{F_{12}}{b^2} \left(\frac{\partial \theta_y}{\partial \eta} \right)^2 \\ & + \frac{4 D_{33}}{a^2 b^2} \left(\frac{\partial^2 w_0}{\partial \xi^2} \frac{\partial^2 w_0}{\partial \eta^2} \right) + \frac{4 H_{33}}{a\, b^2} \left(\frac{\partial^2 w_0}{\partial \xi \partial \eta} \frac{\partial \theta_x}{\partial \eta} \right) + \frac{4 H_{33}}{a^2\, b} \left(\frac{\partial^2 w_0}{\partial \xi \partial \eta} \frac{\partial \theta_y}{\partial \xi} \right) \\ & + \frac{G_{33}}{b^2} \left(\frac{\partial \theta_x}{\partial \eta} \right)^2 + \frac{G_{33}}{a^2} \left(\frac{\partial \theta_y}{\partial \xi} \right)^2 + \frac{2 G_{33}}{a\, b} \left(\frac{\partial \theta_x}{\partial \eta} \frac{\partial \theta_y}{\partial \xi} \right) + \frac{I_{33}}{a^2} \left(\frac{\partial w_0}{\partial \xi} \right)^2 + \frac{2 I_{33}}{a} \left(\frac{\partial w_0}{\partial \xi} \theta_x \right) \\ & + I_{33} \, \theta_x^{\,2} + \frac{I_{44}}{b^2} \left(\frac{\partial w_0}{\partial \eta} \right)^2 + \frac{2 I_{44}}{b} \left(\frac{\partial w_0}{\partial \eta} \theta_y \right) + I_{44} \, \theta_y^{\,2} \Bigg] d\xi \, d\eta \end{aligned} \tag{11}$$

where

$$D_{i,j} = \int_{-h/2}^{h/2} C_{i,j} \, f(z)^2 \, dz, \quad E_{i,j} = \int_{-h/2}^{h/2} C_{i,j} \left(f(z)^2 - 2\, z\, f(z) \right) dz \tag{12, 13}$$

$$F_{i,j} = \int_{-h/2}^{h/2} C_{i,j}\left(f(z)^2 - 2z\,f(z) + z^2\right)dz \quad , \quad G_{i,j} = \int_{-h/2}^{h/2} C_{i,j}\left(f(z)^2 - 2f(z) + 1\right)dz \qquad (14, 15)$$

$$H_{i,j} = \int_{-h/2}^{h/2} C_{i,j}\left(f(z)^2 - f(z)\right)dz \quad , \quad I_{i,j} = \int_{-h/2}^{h/2} C_{i,j}\left(\left(\frac{\partial f(z)}{\partial z}\right)^2 - 2\frac{\partial f(z)}{\partial z} + 1\right)dz \qquad (16, 17)$$

Where $\xi(= x/a)$ and $\eta(= y/b)$ are the non-dimensional coordinates.

2.1. *Hierarchical finite element formulation*

A fournode rectangular hierarchical finite element with six degrees of freedom per node (w_0, $\partial w_0/\partial x$, $\partial w_0/\partial y$, $\partial w_0/\partial xy$, θ_x, θ_y) is developed on the basis of a third-order plate theory (See Fig. 2). Trigonometric hierarchical functions are used as shape functions. The model requires C^0 continuity for θ_x and θ_y and C^1continuity for w_0.

The displacement and rotations of the rectangular plate p-element are expressed as

$$w_0(\xi,\eta,t) = \sum_{m=1}^{P_w}\sum_{n=1}^{P_w} W_{mn}(t)g_m(\xi)g_n(\eta)$$

$$\theta_x(\xi,\eta,t) = \sum_{m=1}^{P_\theta}\sum_{n=1}^{P_\theta} \theta_{x_{mn}}(t)f_m(\xi)f_n(\eta) \qquad (18)$$

$$\theta_y(\xi,\eta,t) = \sum_{m=1}^{P_\theta}\sum_{n=1}^{P_\theta} \theta_{y_{mn}}(t)f_m(\xi)f_n(\eta)$$

Where P_w and P_θ are the number of shape functions used in the model.

The firsts shape functions (f_1 f_2 and g_1 to g_4) are commonly used in the finite element method. The functions (f_{n+2} and g_{n+4})are the trigonometric shape functions and lead to zero transverse displacement, and zero slope at each node. This feature is highly significant since these functions only give additional freedom to the edges and the interior of the element.

The trigonometric hierarchical shape functions $f_i(\xi)$ for C^0 continuity and $g_i(\xi)$for C^1 continuity are given by [24]

$$\begin{cases} f_1 = 1-\xi \\ f_2 = \xi \\ f_{n+2} = \sin(\delta r\,\xi) \\ \delta r = r\,\pi \\ r = 1,2,3,... \end{cases} \qquad (19)$$

and

$$\begin{cases} g_1 = 1 - 3\xi^2 + 2\xi^3 \\ g_2 = \xi - 2\xi^2 + \xi^3 \\ g_3 = 3\xi^2 - 2\xi^3 \\ g_4 = -\xi^2 + \xi^3 \\ g_{n+4} = \delta r \left[-\xi + \left(2 + (-1)^r\right)\xi^2 - \left(1 + (-1)^r\right)\xi^3 \right] + \sin(\delta r \xi) \\ \delta r = r\pi \\ r = 1,2,3,\ldots \end{cases} \quad (20)$$

The displacement and rotationscan be expressed in matrix form

$$\begin{Bmatrix} w_0 \\ \theta_x \\ \theta_y \end{Bmatrix} = [N]\{q\} \quad (21)$$

[N] is the matrix of the shape functions, given by

$$[N] = \begin{bmatrix} [N_w] & 0 & 0 \\ 0 & [N_\theta] & 0 \\ 0 & 0 & [N_\theta] \end{bmatrix} \quad (22)$$

where

$$\{q\} = \begin{Bmatrix} q_w \\ q_{\theta_x} \\ q_{\theta_y} \end{Bmatrix} \quad (23)$$

In which q_w, q_{θ_x} and q_{θ_y} are the generalized displacements.

The matrices of theshape functions are given by

$$[N_w] = \left[\left(g_1(\xi)\ g_1(\eta)\right)_1, \left(g_1(\xi)\ g_2(\eta)\right)_2, \ldots \left(g_k(\xi)\ g_l(\eta)\right)_r, \ldots \left(g_{P_w}(\xi)\ g_{P_w}(\eta)\right)_{P_w P_w} \right] \quad (24)$$

where $k = 1,\ldots,P_w$, $l = 1,\ldots,P_w$, and $r = j + (i-1)P_w$

and

$$[N_\theta]=\Big[\big(f_1(\xi)\ f_1(\eta)\big)_1,\big(f_1(\xi)\ f_2(\eta)\big)_2,\ldots\big(f_i(\xi)\ f_j(\eta)\big)_m,\ldots\big(f_{P_\theta}(\xi)\ f_{P_\theta}(\eta)\big)_{P_\theta P_\theta}\Big] \tag{25}$$

where $i=1,\ldots,P_\theta$, $j=1,\ldots,P_\theta$,and $m=\ j+(i-1)P_\theta$.

The discretized system of equations of bending of free vibration of isotropic plate can be expressed as

$$[M]\{\ddot{q}\}+[K]\{q\}=0 \tag{26}$$

Here [K] is the stiffness matrix of the p-element, determined from the strain energy

$$[K]=\begin{bmatrix}[K_{ww}] & [K_{\mathrm{w\theta x}}] & [K_{\mathrm{w\theta y}}]\\ [K_{\mathrm{w\theta x}}]^T & [K_{\mathrm{\theta x\theta x}}] & [K_{\mathrm{\theta x\theta y}}]\\ [K_{\mathrm{w\theta y}}]^T & [K_{\mathrm{\theta x\theta y}}]^T & [K_{\mathrm{\theta y\theta y}}]\end{bmatrix} \tag{27}$$

Where

$$\begin{aligned}[K_{ww}]=a\,b\int_0^1\!\!\int_0^1\Bigg[&\frac{D_{11}}{a^4}\frac{\partial^2[N_w]^T}{\partial\xi^2}\frac{\partial^2[N_w]}{\partial\xi^2}+\frac{2D_{12}}{a^2\,b^2}\frac{\partial^2[N_w]^T}{\partial\xi^2}\frac{\partial^2[N_w]}{\partial\eta^2}+\frac{D_{22}}{b^4}\frac{\partial^2[N_w]^T}{\partial\eta^2}\frac{\partial^2[N_w]}{\partial\eta^2}\\ &+\frac{4D_{33}}{a^2\,b^2}\frac{\partial[N_w]^T}{\partial\xi\,\partial\eta}\frac{\partial[N_w]}{\partial\xi\,\partial\eta}+\frac{F_{44}}{a^2\,b^2}\frac{\partial[N_w]^T}{\partial\xi}\frac{\partial[N_w]}{\partial\xi}+\frac{F_{44}}{a^2\,b^2}\frac{\partial[N_w]^T}{\partial\eta}\frac{\partial[N_w]}{\partial\eta}\Bigg]d\xi\,d\eta\end{aligned} \tag{28}$$

$$\begin{aligned}[K_{\mathrm{w\theta x}}]=a\,b\int_0^1\!\!\int_0^1\Bigg[&\frac{2E_{11}}{a^2}\frac{\partial^2[N_w]^T}{\partial\xi^2}\frac{\partial[N_\theta]}{\partial\xi}+\frac{2E_{12}}{a^2\,b}\frac{\partial^2[N_w]^T}{\partial\eta^2}\frac{\partial[N_\theta]}{\partial\xi}+\frac{2E_{33}}{a\,b^2}\frac{\partial^2[N_w]^T}{\partial\xi\,\partial\eta}\frac{\partial[N_\theta]}{\partial\eta}\\ &+\frac{2F_{44}}{a}\frac{\partial[N_w]^T}{\partial\xi}[N_\theta]\Bigg]d\xi\,d\eta\end{aligned} \tag{29}$$

$$\begin{aligned}[K_{\mathrm{w\theta y}}]=a\,b\int_0^1\!\!\int_0^1\Bigg[&\frac{2E_{12}}{a^2\,b}\frac{\partial^2[N_w]^T}{\partial\xi^2}\frac{\partial[N_\theta]}{\partial\eta}+\frac{2E_{22}}{b^3}\frac{\partial^2[N_w]^T}{\partial\eta^2}\frac{\partial[N_\theta]}{\partial\eta}+\frac{4E_{33}}{a^2\,b}\frac{\partial^2[N_w]^T}{\partial\xi\,\partial\eta}\frac{\partial[N_\theta]}{\partial\xi}\\ &+\frac{2F_{55}}{a}\frac{\partial[N_w]^T}{\partial\eta}[N_\theta]\Bigg]d\xi\,d\eta\end{aligned} \tag{30}$$

$$\left[K_{\theta x\theta x}\right] = a\,b\int_0^1\int_0^1\left[\frac{G_{11}}{a^2}\frac{\partial\left[N_\theta\right]^T}{\partial\xi}\frac{\partial\left[N_\theta\right]}{\partial\xi}+\frac{G_{33}}{b^2}\frac{\partial\left[N_\theta\right]^T}{\partial\eta}\frac{\partial\left[N_\theta\right]}{\partial\eta}+F_{44}\left[N_\theta\right]^T\left[N_\theta\right]\right]d\xi\,d\eta \tag{31}$$

$$\left[K_{\theta y\theta y}\right] = a\,b\int_0^1\int_0^1\left[\frac{G_{33}}{a^2}\frac{\partial\left[N_\theta\right]^T}{\partial\xi}\frac{\partial\left[N_\theta\right]}{\partial\xi}+\frac{G_{22}}{b^2}\frac{\partial\left[N_\theta\right]^T}{\partial\eta}\frac{\partial\left[N_\theta\right]}{\partial\eta}+F_{44}\left[N_\theta\right]^T\left[N_\theta\right]\right]d\xi\,d\eta \tag{32}$$

$$\left[K_{\theta x\theta y}\right] = a\,b\int_0^1\int_0^1\left[\frac{2G_{12}}{a\,b}\frac{\partial\left[N_\theta\right]^T}{\partial\xi}\frac{\partial\left[N_\theta\right]}{\partial\eta}+\frac{2G_{33}}{a\,b}\frac{\partial\left[N_\theta\right]^T}{\partial\eta}\frac{\partial\left[N_\theta\right]}{\partial\xi}\right]d\xi\,d\eta \tag{33}$$

[M]is themass matrix of the p-element,given by the following relation

$$[M] = \begin{bmatrix} [M_{ww}] & [M_{w\theta x}] & [M_{w\theta y}] \\ [M_{w\theta x}]^T & [M_{\theta x\theta x}] & [M_{\theta x\theta y}] \\ [M_{w\theta y}]^T & [M_{\theta x\theta y}]^T & [M_{\theta y\theta y}] \end{bmatrix} \tag{34}$$

Where

$$\left[M_{ww}\right] = a\,b\int_0^1\int_0^1\left[\rho_A\left[N_w\right]^T\left[N_w\right]+\frac{\rho_E}{a^2}\frac{\partial\left[N_w\right]^T}{\partial\xi}\frac{\partial\left[N_w\right]}{\partial\xi}+\frac{\rho_E}{b^2}\frac{\partial\left[N_w\right]^T}{\partial\eta}\frac{\partial\left[N_w\right]}{\partial\eta}\right]d\xi\,d\eta \tag{35}$$

$$\left[M_{w\theta x}\right] = a\,b\int_0^1\int_0^1\rho_G\frac{\partial\left[N_w\right]^T}{\partial\xi}\left[N_\theta\right]d\xi\,d\eta \tag{36}$$

$$\left[M_{w\theta y}\right] = a\,b\int_0^1\int_0^1\rho_G\frac{\partial\left[N_w\right]^T}{\partial\eta}\left[N_\theta\right]d\xi\,d\eta \tag{37}$$

$$\left[M_{\theta x\theta x}\right] = \left[M_{\theta y\theta y}\right] = a\,b\int_0^1\int_0^1\rho_H\left[N_\theta\right]^T\left[N_\theta\right]d\xi\,d\eta \tag{38}$$

3. Numerical results and discussion

3.1. Convergence study

Tables 1 to 5 show that good convergence and accuracy of the solutions are obtained by increasing the number of trigonometric shape functions, for all cases. It is seen that good results from thick plates are obtained by using, only six shape function in the case of SSSS plates, 14 shape functions in the case CCCC plates,12 shape functions in the case of FFFF plates, and 17 shape functions in others cases.

Table 1.Convergence of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}$ for a thick plates ($\nu = 0.3, a/b = 1, h/a = 0.5$) with SSSS boundary condition.

$P_\theta = P_w$	1	2	3	4	5
4	1.263	2.47	2.47	2.919	2.919
5	1.246	2.414	2.414	2.919	2.919
6	1.245	2.308	2.308	2.919	2.919
Converged solution	1.245	2.308	2.308	2.919	2.919

Table 2.Convergence of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}$ for a thick plates ($\nu = 0.3, a/b = 1, h/a = 0.5$) with CCCC boundary condition.

$P_\theta = P_w$	1	2	3	4	5
4	3.623	3.623	4.026	4.557	5.069
5	1.662	3.617	3.617	3.965	4.515
6	1.659	2.639	2.639	3.466	3.695
7	1.618	2.613	2.613	3.448	3.691
8	1.615	2.586	2.586	3.405	3.687
9	1.604	2.575	2.575	3.392	3.686
10	1.601	2.566	2.566	3.379	3.686
11	1.598	2.559	2.559	3.369	3.685
12	1.595	2.556	2.556	3.366	3.685
13	1.595	2.552	2.552	3.358	3.685
14	1.593	2.55	2.55	3.358	3.685
Converged solution	1.593	2.55	2.55	3.358	3.685

Table 3.Convergence of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}$ for a thick plates ($\nu = 0.3, a/b = 1, h/a = 0.5$) with FFFF boundary condition.

$P_\theta = P_w$	1	2	3	4	5
4	0.905	1.354	1.693	1.939	1.939
5	0.905	1.266	1.514	1.805	1.805

6	0.895	1.266	1.514	1.798	1.798
7	0.895	1.265	1.511	1.793	1.793
8	0.893	1.265	1.511	1.792	1.792
9	0.893	1.265	1.511	1.79	1.790
10	0.892	1.265	1.511	1.79	1.790
11	0.892	1.265	1.511	1.789	1.789
12	0.892	1.264	1.511	1.789	1.789
Converged solution	0.892	1.264	1.511	1.789	1.789

Table 4. Convergence of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}$ for a thick plates($\nu = 0.3, a/b = 1, h/a = 0.5$) with CFCF boundary condition.

$P_\theta = P_w$	1	2	3	4	5
4	2.883	3.378	3.45	3.56	3.792
5	1.139	1.248	1.947	2.721	3.369
6	1.137	1.241	1.944	2.261	2.450
7	1.114	1.213	1.913	2.252	2.439
8	1.112	1.209	1.91	2.226	2.411
9	1.106	1.202	1.903	2.218	2.403
10	1.103	1.199	1.900	2.210	2.395
11	1.102	1.197	1.898	2.204	2.389
12	1.100	1.195	1.897	2.202	2.387
13	1.100	1.195	1.896	2.198	2.382
14	1.099	1.193	1.895	2.198	2.382
15	1.099	1.193	1.895	2.194	2.379
16	1.098	1.192	1.894	2.194	2.378
17	1.098	1.192	1.894	2.192	2.376
Converged solution	1.098	1.192	1.894	2.192	2.376

Table 5.Convergence of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}$ for a thick plates ($\nu = 0.3, a/b = 1, h/a = 0.5$) with CSCS boundary condition.

$P_\theta = P_w$	1	2	3	4	5
4	3.418	3.499	3.592	4.331	4.432
5	1.656	3.418	3.485	3.585	4.073
6	1.654	2.611	2.635	3.351	3.521

7	1.612	2.580	2.610	3.343	3.517
8	1.609	2.554	2.583	3.304	3.514
9	1.599	2.542	2.572	3.295	3.513
10	1.595	2.533	2.563	3.284	3.513
11	1.593	2.526	2.556	3.276	3.512
12	1.590	2.523	2.553	3.273	3.512
13	1.590	2.519	2.549	3.267	3.512
14	1.587	2.517	2.548	3.267	3.512
15	1.587	2.514	2.545	3.263	3.512
16	1.586	2.514	2.544	3.263	3.512
17	1.586	2.512	2.542	3.260	3.512
Converged solution	1.586	2.512	2.542	3.260	3.512

3.2. Discussion

The results obtained for an isotropic plate by applying Higher-order using rectangular p-element, are compared with those available in the literature. The linear natural frequencies of plates with free edges (FFFF), simply supported (SSSS), fully clamped plates (CCCC), and combined boundary condition (CFCF), (SFSF) are considered.

The frequency parameter of the plate is expressed as

$$\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho\, h}{D}}, \qquad (36)$$

Where $D = \frac{E\, h^3}{12\,(1-\nu^2)}$ is the flexural rigidity of the plate.

The first five frequency parameters Ω for simply-supported (SSSS) square plates with different thickness-side ratio $h/b = 0.001, 0.1, 0.2, 0.5$ computed using the present method are given in Table 6 and compared with other published solutions, Leissa [17], Houmat [24], 3-D exact solution [18], Nayak [9], The authors in 1956 [2], Lim and his colleagues [18], Srinivas and his colleagues [19], Malik and Bert, 1998 [27], Zhou and his colleagues [20], very good agreement can be observed, in the results with concealment compare of Nayak [9] are obtained with 1245 DOF, or ours are obtained with 480 DOF $P_\theta = P_w = 19$, in the example illustrated in table 6, a very good accuracy is observed.

In table 7, good agreement is achieved by comparing the present results with those obtained by Wang [21], Leissa [17], Lim and his colleagues [1], Liew and his colleagues [23], Wang [22], The first five frequency parameters Ω for fully clamped (CCCC) square plates with different thickness-side ratio $h/b = 0.001, 0.1, 0.2, 0.5$.

Table 6.Comparison of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2} \sqrt{\frac{\rho h}{D}}$ of a SSSS square plate ($\nu = 0.3$).

h/a	Solution methods	1	2	3	4	5
0.001	Present (Pθ=Pw=19)	2,000	5,003	5,003	8,004	10,017
	CPT-exact [17]	2,000	5,000	5,000	8,000	10,000
0.1	Present (480 dofs)	1,932	4,609	4,609	7.074	8.622
	HSDT FEM [9] (1245 dofs)	1,931	4,614	4,614	7.085	8.657
	MindlinTheory [2]	1,931	4,605	4,605	7.064	8.607
	HSDT [18]	1,932	4,609	4,609	7.073	8.617
	3-D exact [19]	1,934	4,622	4,622	7.103	8.662
0.2	Present (Pθ=Pw=12)	1,768	3.870	3.870	5.599	6.619
	MindlinTheory [2]	1,766	3.858	3.858	5.573	-
	3-D exact [19]	1,756	3.899	3.899	5.653	-
0.5	Present (Pθ=Pw=6)	1,245	2.308	2.308	2.919	2.919
	HSDT [18]	1,245	2.308	2.308	2.917	2.917
	3-D exact [19]	1,259	-	-	-	-
	3-D Ritz [20]	1,259	2.331	2.331	-	-

Table 7.Comparison of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2} \sqrt{\frac{\rho h}{D}}$ of a CCCC square plate ($\nu = 0.3$).

h/a	Solution methods	1	2	3	4	5
0.001	Present (Pθ=Pw=24)	3.674	7.506	7.506	11.162	13.488
	CPT Ritz (S wang) [21]	3,646	7,436	7,436	10,945	13,332
	CPT Ritz [17]	3,647	7,438	7,438	10,970	13,338
0.1	Present (Pθ=Pw=22)	3.306	6.323	6.323	8.879	10.477
	HSDT [18]	3,303	6,311	6,311	8,858	10,446
	3-D Ritz [22]	3,322	6,346	6,346	8,903	10,498
0.2	Present (Pθ=Pw=22)	2.720	4.786	4.786	6.451	7.384
	MindlinTheory [23]	2,681	4,675	4,675	-	-
0.5	Present (Pθ=Pw=14)	1.593	2.550	2.550	3.358	3.685

HSDT [18]	1,587	2.536	2.536	3.337	3.684
3-D Ritz [22]	1,550	2,515	2,515	3,193	3.654

In table 8, good agreement is achieved by comparing the present results with those obtained by, Liew and his colleagues . [23] and Lim and his colleagues [18], The first five frequency parameters Ω for combined boundary condition (CFCF) square plates with tree different thickness-side ratio $h/b = 0.01, 0.1, 0.5$.

Table 8.Comparison of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}$ of a CFCF square plate ($\nu = 0.3$).

h/b	Solution methods	1	2	3	4	5
0.01	Present (Pθ=Pw=22)	2.245	2.673	4.410	6.189	6.795
	3-D Ritz [22]	2.248	2.674	4.408	6.197	6.799
0.1	Present (Pθ=Pw=22)	2.095	2.441	3.920	5.366	5.813
	HSDT [18]	2.093	2.438	3.913	5.357	5.802
	3-D Ritz [22]	2.105	2.449	3.923	5.386	5.827
0.5	Present (Pθ=Pw=17)	1.098	1.192	1.894	2.192	2.376
	HSDT [18]	1.095	1.189	1.891	2.185	2.369
	3-D Ritz [22]	1.072	1.193	1.871	2.193	2.325

In table 9, shows a comparison of results for square thin and thick plates by, Houmat [24], and Lim and his colleagues [18], The first five frequency parameters Ω for free edges (FFFF) square plates with different thickness-side ratio $h/b = 0.001, 0.1, 0.5$.

Table 9.Comparison of frequency parameters $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}$ of a FFFF square plate ($\nu = 0.3$).

h/a	Solution methods	1	2	3	4	5
0.001	Present (Pθ=Pw=20)	1,365	1,986	2,460	3,529	6,199
	HFEM [24]	1,365	1,986	2,459	3,526	6,190
0.1	Present (Pθ=Pw=20)	1,291	1,919	2,363	3,239	3,239
	HSDT [18]	1,289	1,919	2,363	3,235	3,235
0.5	Present (Pθ=Pw=12)	0,892	1,264	1,511	1.789	1.789
	HSDT [18]	0,891	1,264	1,511	1.788	1.788

In table 10, good agreement has been achieved. The attained accuracy is also confirmed by comparing our results, Leissa [17], Hosseini and his colleagues [25,26], Malik [27], Liew and his colleagues [23], The first five frequency parameters $\Omega = \omega\, a^2\sqrt{\rho h/D}$ for combined boundary condition (SFSF) square plates with different thickness-side ratio $h/b = 0.001, 0.1, 0.2, 0.5$.

Table 10.Comparison of frequency parameters $\Omega = \omega\, a^2\sqrt{\rho h/D}$ of a SFSF square plate ($\nu = 0.3$).

h/a	Solution methods	1	2	3	4	5
0.001	Present (Pθ=Pw=17)	9.638	16.142	36.730	39.027	46.860
	CPT-exact [17]	9,631	16,135	36,726	38,945	46,738
	HSDT Exact [25]	9,631	16,131	36,716	38,943	46,732
0.1	Present (Pθ=Pw=20)	9,442	15,401	33,896	36,367	42,821
	Exact FSDT [26]	9,446	15,405	33,916	36,425	42,887
	3-D DQM [27]	9,446	15,400	33,911	36,437	42,887
	3-D Ritz [22]	9,446	15,400	33,913	36,438	42,887
	HSDT Exact [25]	9,446	15,392	33,868	36,349	42,801
0.2	Present (Pθ=Pw=20)	8,984	14,104	29,170	31,296	36,006
	Exact FSDT [26]	9,000	14,134	29,256	31,434	36,165
	3-D DQM [27]	9,001	14,123	29,263	31,472	36,173
	3-D Ritz [22]	9,001	14,122	29,263	31,472	36,173
	HSDT Exact [25]	8,984	14,101	29,162	31,293	35,999
0.5	Present (Pθ=Pw=20)	7,121	10,070	18,196	19,470	21,628
	3-D Ritz [22]	7,166	10,100	18,210	19,613	21,652
	HSDT Exact [25]	7,121	10,069	18,194	19,469	21,626

In table 11 and 12, in has to study the effect of the boundary conditions, thickness ratio and rapport a/b on frequency parameters Ω, These tabulated frequency parameters should be useful as benchmark solutions for researchers who are developing numerical techniques and software for solving isotropic HSDT plate vibration problems.

Table 11.Frequency parameter $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho\, h}{D}}$ $(\nu = 0.3)$ of FFFF, CCCC and CFCF plates.

B.C	a/b	h/a	1	2	3	4	5
FFFF	0.5	0.1	2.096	2.439	5.172	5.466	7.850
		0.2	1.917	2.090	4.219	4.477	6.220
		0.5	1.201	1.365	2.278	2.408	2.772
		0.1	1.291	1.919	2.363	3.239	3.239
	1	0.2	1.187	1.763	2.148	2.798	2.798
		0.5	0.892	1.264	1.511	1.789	1.789
		0.1	0.865	0.948	1.963	2.166	2.463
	1.5	0.2	0.809	0.906	1.766	1.978	2.196
		0.5	0.630	0.731	1.243	1.402	1.451
		0.1	0.538	0.646	1.409	1.467	2.149
	2	0.2	0.524	0.609	1.293	1.367	1.962
		0.5	0.538	0.646	1.409	1.467	2.149
	0.5	0.1	7.872	9.913	13.323	17.15	17.739
		0.2	5.569	6.925	9.039	10.885	11.631
		0.5	2.863	3.510	4.331	4.541	4.959
		0.1	3.306	6.323	6.323	8.879	10.477
	1	0.2	2.720	4.786	4.786	6.451	7.384
		0.5	1.593	2.550	2.550	3.358	3.685
		0.1	2.531	3.811	5.767	5.858	6.831
	1.5	0.2	2.140	3.111	4.407	4.553	5.149
		0.5	1.303	1.804	2.363	2.508	2.755
		0.1	2.314	2.960	4.086	5.599	5.632
	2	0.2	1.968	2.478	3.331	4.287	4.435
		0.5	1.203	1.493	1.939	2.298	2.476
	0.5	0.1	2.080	3.152	5.328	6.852	9.214
		0.2	1.782	2.473	4.098	5.074	6.851
		0.5	1.095	1.253	2.186	2.513	3.434
		0.1	2.095	2.441	3.920	5.366	5.813
		0.2	1.793	2.036	3.194	4.120	4.430
	1						
CFCF		0.5	1.098	1.192	1.894	2.192	2.376

1.5	0.1	2.101	2.264	2.907	4.225	5.381
	0.2	1.798	1.915	2.41	3.446	4.13
	0.5	1.099	1.150	1.443	1.990	2.198
2	0.1	2.105	2.198	2.555	3.265	4.400
	0.2	1.800	1.867	2.140	2.703	3.589
	0.5	1.100	1.131	1.285	1.619	2.084

Table 12.Frequency parameter $\Omega = \omega \frac{b^2}{\pi^2}\sqrt{\frac{\rho h}{D}}$ $(\nu = 0.3)$ of CFFF, CSCS and SFSF plates.

B.C	a/b	h/a	1	2	3	4	5
CFFF	0.5	0.1	1.367	1.997	3.658	6.611	7.332
		0.2	1.257	1.730	3.018	5.120	5.321
		0.5	0.908	1.120	1.834	2.339	2.499
	1	0.1	0.348	0.818	2.038	2.586	2.868
		0.2	0.339	0.747	1.790	2.282	2.435
		0.5	0.296	0.531	1.125	1.502	1.523
	1.5	0.1	0.155	0.502	0.941	1.666	2.294
		0.2	0.152	0.467	0.879	1.490	2.057
		0.5	0.142	0.346	0.649	1.023	1.380
	2	0.1	0.087	0.360	0.534	1.161	1.471
		0.2	0.086	0.337	0.513	1.062	1.349
		0.5	0.082	0.256	0.417	0.766	0.958
CSCS	0.5	0.1	7.867	9.898	13.297	17.142	17.706
		0.2	5.565	6.913	9.023	10.87	11.616
		0.5	2.859	3.504	4.319	4.535	4.898
	1	0.1	3.302	6.312	6.314	8.854	10.461
		0.2	2.710	4.769	4.770	6.412	7.358
		0.5	1.586	2.512	2.542	3.260	3.512
	1.5	0.1	2.528	3.804	5.762	5.848	6.814
		0.2	2.132	3.093	4.395	4.533	5.115
		0.5	1.295	1.779	2.359	2.491	2.728
	2	0.1	2.313	2.955	4.078	5.596	5.622
		0.2	1.962	2.462	3.310	4.279	4.413
		0.5	1.194	1.470	1.917	2.294	2.453
SFSF	0.5	0.1	0.945	2.527	3.641	5.716	7.698
		0.2	0.900	2.155	3.137	4.575	6.006
		0.5	0.718	1.202	1.963	2.414	2.611
	1	0.1	0.957	1.560	3.434	3.685	4.338
		0.2	0.910	1.429	2.955	3.171	3.648
		0.5	0.722	1.020	1.844	1.973	2.191
	1.5	0.1	0.964	1.274	2.204	3.702	3.777
		0.2	0.917	1.188	1.977	3.185	3.225
		0.5	0.726	0.892	1.343	1.945	1.98
	2	0.1	0.968	1.153	1.721	2.655	3.711
		0.2	0.921	1.085	1.576	2.349	3.193
		0.5	0.729	0.831	1.124	1.540	1.984

4. Conclusion

An hierarchical finite element formulation is presented based on an Higher-order shear plates theory for homogeneous thick plates, a computer program is developed, A p-version, hierarchical finite element was presented and applied to thick plate with trigonometric hierarchical shape functions.

1- Based on the Reddy's higher-order theory, these elements have been implemented with a very simple and understandable mathematical framework and are easily programmed.
2- High accuracy, stable numerical computation and rapid convergence have been observed in the analysis.
3- The results are compared with other shear deformation theories and exact elasticity solutions. Good agreement may be noted.

Several examples are solved. The effect of boundary condition and ratio a/b and h/a the analyzes is also studied.

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