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On the Uniquness of Solutions of Linear Ordinary Fractional Differential Equations by Using Different Integral Transform Methods

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Abstract

The main objective of this paper has to investigate the uniqueness of the solution of fractional differential equation by using different integral transforms, we applied Laplace transform, Elzaki transform and Sumudu transform on a linear ordinary fractional differential equation. The uniqueness of the solution is achieved in fractional differential equations by applying different integral transform methods.

Keywords: fractional differential equation; Laplace transform; Elzaki transform; Sumudu transform.

1. Introduction

The Fractional Calculus which is the branch of mathematics remained inactive from the 17th century to early 20th century. Fractional Calculus deals with derivative and integrals of arbitrary order, since last three decades the fractional calculus found applications in various areas of studies in applied mathematics and science like fluid flow, rheology, diffusion, oscillation, anomalous diffusion, reaction-diffusion, turbulence, electric network, physics, chemistry, waves, dynamical problems and statistical distribution theory.

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Now a day's mathematicians and researcher working with fractional differential equations and discover numerous applications in the areas of applied mathematics, engineering and physics [1]. Various important phenomena in many fields of science such as electromagnetics, acoustics, viscoelasticity, electrochemistry, and material science are discussed by fractional differential equations (FDE) [2–4]. Also, fractional differential equations have been found to be effective to describe some. The physical phenomena such as damping laws, rheology, diffusion processes are also well described by fractional differential equations. There are many methods have been used to solve the fractional differential equations, such as Adomian's decomposition method (ADM) [5-7], Fourier transform method [8], Laplace transform method [2,3,9], and so on.

2. Results and Discussions

Let us consider a linear fractional differential equation

$$D^{\frac{\beta}{2}}u(t)=\frac{\Gamma(2m+3)}{\Gamma\left(\frac{\beta}{2}+2\right)}t^{\frac{\beta}{2}+1}-\frac{\Gamma(2m+2)}{\Gamma\left(\frac{\beta}{2}+1\right)}t^{\frac{\beta}{2}}+\frac{\Gamma(2m+1)}{\Gamma\left(\frac{\beta}{2}\right)}t^{\frac{\beta}{2}-1}+t^2-t+1 \qquad (1)$$

where $\beta=1,3,5,7,\dots 2n-1$ *and* $m=0,1,2,3,4,\dots$

Let us discuss the solution of equation (1) by using Laplace, Sumudu and Elzaki transforms to solve equation (1)

2.1 Solution by using Laplace transform method

Laplace transform is an extremely useful method for solving linear ODEs and related initial value problems, as well as systems of linear ODEs, much easier. Fundamental formulas and definitions are given in [10]. In order to get solution of equation (1), we first consider $\beta=1$ and $m=0$ **,** the equation (1) takes the form

$$D^{1/2}u(t)=\frac{\Gamma(3)}{\Gamma(5/2)}t^{3/2}-\frac{\Gamma(2)}{\Gamma(3/2)}t^{1/2}+\frac{\Gamma(1)}{\Gamma(1/2)}t^{-1/2}+t^2-t+1 \qquad (2)$$

with subject to initial condition $u(0)=0$

Applying the Laplace transform on equation (2), we get

$$\frac{sU(s)-u(0)}{s^{1-1/2}}=\frac{\Gamma(3)}{s^{5/2}}-\frac{\Gamma(2)}{s^{3/2}}+\frac{\Gamma(1)}{s^{1/2}}+\frac{2!}{s^3}-\frac{1}{s^2}+\frac{1}{s} \qquad (3)$$

using $u(0)=0$, we obtain

$$\frac{sU(s)}{s^{-1/2}}=\frac{\Gamma(3)}{s^{5/2}}-\frac{\Gamma(2)}{s^{3/2}}+\frac{\Gamma(1)}{s^{1/2}}+\frac{2!}{s^3}-\frac{1}{s^2}+\frac{1}{s}$$

$$s^{1/2}U(s) = \frac{\Gamma(3)}{s^{5/2}} - \frac{\Gamma(2)}{s^{3/2}} + \frac{\Gamma(1)}{s^{1/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

Or $$U(s) = \frac{\Gamma(3)}{s^3} - \frac{\Gamma(2)}{s^2} + \frac{\Gamma(1)}{s} + \frac{2!}{s^{7/2}} - \frac{1}{s^{5/2}} + \frac{1}{s^{3/2}} \tag{4}$$

Now using inverse Laplace transform we get the solution

$$u(t) = t^2 - t + 1 + \frac{2t^{5/2}}{\Gamma(7/2)} - \frac{t^{3/2}}{\Gamma(5/2)} + \frac{t^{1/2}}{\Gamma(3/2)} \tag{5}$$

Now For $\beta = 3$ and $m = 1$ equation (1), takes the form as follows

$$D^{3/2}u(t) = \frac{\Gamma(5)}{\Gamma(7/2)}t^{5/2} - \frac{\Gamma(4)}{\Gamma(5/2)}t^{3/2} + \frac{\Gamma(3)}{\Gamma(3/2)}t^{1/2} + t^2 - t + 1 \tag{6}$$

Subject to initial condition $U(0) = U'(0) = 0$

Applying the Laplace transform on equation $(6), U(s)$ is obtained as follows

$$\frac{s^2U(s) - su(0) - u'(0)}{s^{2-3/2}} = \frac{\Gamma(5)}{s^{7/2}} - \frac{\Gamma(4)}{s^{5/2}} + \frac{\Gamma(3)}{s^{3/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

Applying the initial condition

$$\frac{s^2U(s)}{s^{1/2}} = \frac{\Gamma(5)}{s^{7/2}} - \frac{\Gamma(4)}{s^{5/2}} + \frac{\Gamma(3)}{s^{3/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

$$s^{3/2}U(s) = \frac{\Gamma(5)}{s^{7/2}} - \frac{\Gamma(4)}{s^{5/2}} + \frac{\Gamma(3)}{s^{3/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

$$U(s) = \frac{\Gamma(5)}{s^5} - \frac{\Gamma(4)}{s^4} + \frac{\Gamma(3)}{s^3} + \frac{2!}{s^{9/2}} - \frac{1}{s^{7/2}} + \frac{1}{s^{5/2}}$$

Using inverse Laplace Transform $u(t)$ is obtained as follows

$$u(t) = t^4 - t^3 + t^2 + \frac{2t^{7/2}}{\Gamma(9/2)} - \frac{t^{5/2}}{\Gamma(7/2)} + \frac{t^{3/2}}{\Gamma(5/2)}$$

$$u(t) = t^4 - t^3 + t^2 + \frac{2t^{7/2}}{\Gamma(9/2)} - \frac{t^{5/2}}{\Gamma(7/2)} + \frac{t^{3/2}}{\Gamma(5/2)} \tag{7}$$

For $\beta = 5$ and $n = 2$ equation (1),takes the form as follows

$$D^{5/2}u(t) = \frac{\Gamma(7)}{\Gamma(9/2)}t^{7/2} - \frac{\Gamma(6)}{\Gamma(7/2)}t^{5/2} + \frac{\Gamma(5)}{\Gamma(5/2)}t^{3/2} + t^2 - t + 1 \tag{8}$$

subject to initial condition $u(0) = u'(0) = u''(0) = 0$

Applying the Laplace transform on equation $(4), U(s)$ is obtained as follows

$$\frac{s^3U(s) - s^2u(0) - su(0) - u(0)}{s^{3-5/2}} = \frac{\Gamma(7)}{s^{9/2}} - \frac{\Gamma(6)}{s^{7/2}} + \frac{\Gamma(5)}{s^{5/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

applying the initial condition

$$\frac{s^3U(s)}{s^{1/2}} = \frac{\Gamma(7)}{s^{9/2}} - \frac{\Gamma(6)}{s^{7/2}} + \frac{\Gamma(5)}{s^{5/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

$$s^{5/2}U(s) = \frac{\Gamma(7)}{s^{9/2}} - \frac{\Gamma(6)}{s^{7/2}} + \frac{\Gamma(5)}{s^{5/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

$$U(s) = \frac{\Gamma(7)}{s^7} - \frac{\Gamma(6)}{s^6} + \frac{\Gamma(5)}{s^5} + \frac{2!}{s^{11/2}} - \frac{1}{s^{9/2}} + \frac{1}{s^{7/2}}$$

using inverse Laplace Transform $u(t)$ is obtained as follows

$$u(t) = t^6 - t^5 + t^4 + \frac{2t^{9/2}}{\Gamma(11/2)} - \frac{t^{7/2}}{\Gamma(9/2)} + \frac{t^{5/2}}{\Gamma(7/2)} \tag{9}$$

For $\beta = 7$ and $n = 3$ equation (1) takes the form

$$D^{7/2}U(t) = \frac{\Gamma(9)}{\Gamma(11/2)}t^{9/2} - \frac{\Gamma(8)}{\Gamma(9/2)}t^{7/2} + \frac{\Gamma(7)}{\Gamma(7/2)}t^{5/2} + t^2 - t + 1 \tag{10}$$

subject to initial condition $u(0) = u'(0) = u''(0) = u'''(0) = 0$

using the Laplace transform on equation $(10), U(s)$ is obtained as follows

$$\frac{s^5U(s) - s^4u(0) - s^3u(0) - s^2u(0) - su(0) - u(0)}{s^{5-7/2}} = \frac{\Gamma(9)}{s^{11/2}} - \frac{\Gamma(8)}{s^{9/2}} + \frac{\Gamma(7)}{s^{7/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

using the initial conditions, we obtain

$$\frac{s^5U(s)}{s^{3/2}} = \frac{\Gamma(9)}{s^{11/2}} - \frac{\Gamma(8)}{s^{9/2}} + \frac{\Gamma(7)}{s^{7/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

$$s^{7/2}U(s) = \frac{\Gamma(9)}{s^{11/2}} - \frac{\Gamma(8)}{s^{9/2}} + \frac{\Gamma(7)}{s^{7/2}} + \frac{2!}{s^3} - \frac{1}{s^2} + \frac{1}{s}$$

$$U(s) = \frac{\Gamma(9)}{s^9} - \frac{\Gamma(8)}{s^8} + \frac{\Gamma(7)}{s^7} + \frac{2!}{s^{13/2}} - \frac{1}{s^{11/2}} + \frac{1}{s^{9/2}}$$

Using inverse Laplace transform $u(t)$ is obtained as follows

$$u(t) = t^8 - t^7 + t^6 + \frac{2t^{11/2}}{\Gamma(13/2)} - \frac{t^{9/2}}{\Gamma(11/2)} + \frac{t^{7/2}}{\Gamma(9/2)} \tag{11}$$

Similarly, for $\beta = 9$ and $n = 4$, equation (1) takes the form

$$D^{9/2}U(t) = \frac{\Gamma(11)}{\Gamma(13/2)} t^{11/2} - \frac{\Gamma(10)}{\Gamma(11/2)} t^{9/2} + \frac{\Gamma(9)}{\Gamma(9/2)} t^{7/2} + t^2 - t + 1 \tag{12}$$

applying the Laplace transform on (11) the obtained results is as follows

$$U(t) = t^{10} - t^9 + t^8 + \frac{2t^{13/2}}{\Gamma(15/2)} - \frac{t^{11/2}}{\Gamma(13/2)} + \frac{t^{9/2}}{\Gamma(11/2)} \tag{13}$$

similarly, for $\beta = 2n + 1$ and $m = n$, equation (1), takes the form

$$D^{2n+1/2}u(t) = \frac{\Gamma(2n+3)}{\Gamma\left(2n+1/2+2\right)} t^{2n+1/2+1} - \frac{\Gamma(2n+2)}{\Gamma\left(2n+1/2+1\right)} t^{2n+1/2} + \frac{\Gamma(2n+1)}{\Gamma\left(2n+1/2\right)} t^{2n+1/2-1} + t^2 - t + 1 \tag{14}$$

combining results of (5), (7). (9) and (11) we can generalize the solution for all cases

$$u(t) = t^{2n+2} - t^{2n+1} + t^{2n} + \frac{2t^{\beta+2}}{\Gamma(\beta+3)} - \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta}}{\Gamma(\beta+1)} \tag{15}$$

2.2 Solution by using Sumudu transform method

The Sumudu transform method (STM) was introduced in 1993, by Watugala [11-,12]

Taking Sumudu transform on equation (2), $T(u)$ is obtained as follows

$$\frac{T(u)}{u^{1/2}} - \frac{D^{-1/2}u(t)}{u}\Big|_{t=0} = \Gamma(3)u^{3/2} - \Gamma(2)u^{1/2} + \Gamma(1)u^{-1/2} + 2u^2 - u + 1$$

Using $u(0) = 0$, we get

$$\frac{T(u)}{u^{1/2}} = \Gamma(3)u^{3/2} - \Gamma(2)u^{1/2} + \Gamma(1)u^{-1/2} + 2u^2 - u + 1$$

$$T(u) = \Gamma(3)u^2 - \Gamma(2)u + \Gamma(1) + 2u^{5/2} - u^{3/2} + u^{1/2}$$

using inverse Sumudu transform $u(t)$ is obtained as follows

$$u(t) = t^2 - t + 1 + \frac{2t^{5/2}}{\Gamma(7/2)} - \frac{t^{3/2}}{\Gamma(5/2)} + \frac{t^{1/2}}{\Gamma(3/2)} \tag{16}$$

using the Sumudu transform on equation $(6), T(u)$ is obtained as follows

$$\frac{T(u)}{u^{3/2}} - \frac{D^{1/2}u(t)}{u}|_{t=0} = \Gamma(5)u^{5/2} - \Gamma(4)u^{3/2} + \Gamma(3)u^{1/2} + 2u^2 - u + 1$$

using the initial conditions

$$\frac{T(u)}{u^{3/2}} = \Gamma(5)u^{5/2} - \Gamma(4)u^{3/2} + \Gamma(3)u^{1/2} + 2u^2 - u + 1$$

$$T(u) = \Gamma(5)u^4 - \Gamma(4)u^3 + \Gamma(3)u^2 + 2u^{7/2} - u^{5/2} + u^{3/2}$$

using inverse Sumudu transform on equation, $u(t)$ is obtained as follows

$$u(t) = t^4 - t^3 + t^2 + \frac{2t^{7/2}}{\Gamma(9/2)} - \frac{t^{5/2}}{\Gamma(7/2)} + \frac{t^{3/2}}{\Gamma(5/2)}$$

using the Sumudu transform on equation $(8), T(u)$ is obtained as follows

$$\frac{T(u)}{u^{5/2}} - \frac{D^{3/2}u(t)}{u}|_{t=0} = \Gamma(7)u^{7/2} - \Gamma(6)u^{5/2} + \Gamma(5)u^{3/2} + 2u^2 - u + 1$$

Applying the initial conditions

$$\frac{T(u)}{u^{5/2}} = \Gamma(7)u^{7/2} - \Gamma(6)u^{5/2} + \Gamma(5)u^{3/2} + 2u^2 - u + 1$$

$$T(u) = \Gamma(7)u^6 - \Gamma(6)u^5 + \Gamma(5)u^4 + 2u^{9/2} - u^{7/2} + u^{5/2}$$

using inverse Sumudu transform $u(t)$ is obtained as follows

$$u(t) = t^6 - t^5 + t^4 + \frac{2t^{9/2}}{\Gamma(11/2)} - \frac{t^{7/2}}{\Gamma(9/2)} + \frac{t^{5/2}}{\Gamma(7/2)} \tag{17}$$

Using the Sumudu Transform on equation $(10), T(u)$ is obtained as follows

$$\frac{T(u)}{u^{7/2}} - \frac{D^{5/2}u(t)}{u}|_{t=0} = \Gamma(9)u^{9/2} - \Gamma(8)u^{7/2} + \Gamma(7)u^{5/2} + 2u^2 - u + 1$$

Applying the initial conditions

$$\frac{T(u)}{u^{7/2}} = \Gamma(9)u^{9/2} - \Gamma(8)u^{7/2} + \Gamma(7)u^{5/2} + 2u^2 - u + 1$$

$$T(u) = \Gamma(9)u^8 - \Gamma(8)u^7 + \Gamma(7)u^6 + 2u^{11/2} - u^{9/2} + u^{7/2}$$

using inverse Sumudu transform $u(t)$ is obtained as follows

$$u(t) = t^8 - t^7 + t^6 + \frac{2t^{11/2}}{\Gamma(13/2)} - \frac{t^{9/2}}{\Gamma(11/2)} + \frac{t^{7/2}}{\Gamma(9/2)} \tag{18}$$

In a similar way as for Laplace transform case, the solution of equation (14) also found as

$$u(t) = t^{2n+2} - t^{2n+1} + t^{2n} + \frac{2t^{\beta+2}}{\Gamma(\beta+3)} - \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta}}{\Gamma(\beta+1)} \tag{19}$$

2.3 Solution by using Elzaki transform

The Elzaki transform was introduced by Elzaki [13], using Elzaki transform on equation (3), $E(u)$ is obtained as follows

$$\frac{E(u)}{u^{3/2}} - \frac{D^{1/2}}{u}|_{t=0} = \Gamma(5)u^{9/2} - \Gamma(4)u^{7/2} + \Gamma(3)u^{5/2} + 2u^4 - u^3 + u^2$$

Applying the initial conditions

$$\frac{E(u)}{u^{3/2}} = \Gamma(5)u^{9/2} - \Gamma(4)u^{7/2} + \Gamma(3)u^{5/2} + 2u^4 - u^3 + u^2$$

$$E(u) = \Gamma(5)u^6 - \Gamma(4)u^5 + \Gamma(3)u^4 + 2u^{11/2} - u^{9/2} + u^{7/2}$$

Using inverse Elzaki Transform $u(t)$ is obtained as follows

$$u(t) = t^4 - t^3 + t^2 + \frac{2t^{7/2}}{\Gamma(9/2)} - \frac{t^{5/2}}{\Gamma(7/2)} + \frac{t^{3/2}}{\Gamma(5/2)} \tag{20}$$

applying Elzaki transform on equation (6), $E(u)$ is obtained as follows

$$\frac{E(u)}{u^{5/2}} - \frac{D^{3/2}}{u}|_{t=0} = \Gamma(7)u^{11/2} - \Gamma(6)u^{9/2} + \Gamma(5)u^{7/2} + 2u^4 - u^3 + u^2$$

applying the initial conditions

$$\frac{E(u)}{u^{5/2}} = \Gamma(7)u^{11/2} - \Gamma(6)u^{9/2} + \Gamma(5)u^{7/2} + 2u^4 - u^3 + u^2$$

$$E(u) = \Gamma(7)u^8 - \Gamma(6)u^7 + \Gamma(5)u^6 + 2u^{13/2} - u^{11/2} + u^{9/2}$$

Using the inverse Elzaki Transform $U(t)$ is obtained as follows

$$u(t) = t^6 - t^5 + t^4 + \frac{2t^{9/2}}{\Gamma(11/2)} - \frac{t^{7/2}}{\Gamma(9/2)} + \frac{t^{5/2}}{\Gamma(7/2)} \qquad (21)$$

applying Elzaki transform on equation $(5), T(u)$ is obtained as follows

$$\frac{T(u)}{u^{7/2}} - \frac{D^{5/2}}{u}|_{t=0} = \Gamma(9)u^{13/2} - \Gamma(8)u^{11/2} + \Gamma(7)u^{9/2} + 2u^4 - u^3 + u^2$$

using the initial conditions

$$\frac{E(u)}{u^{7/2}} = \Gamma(9)u^{13/2} - \Gamma(8)u^{11/2} + \Gamma(7)u^{9/2} + 2u^4 - u^3 + u^2$$

$$E(u) = \Gamma(9)u^{10} - \Gamma(8)u^9 + \Gamma(7)u^8 + 2u^{15/2} - u^{13/2} + u^{11/2}$$

Using the inverse Elzaki transform $U(t)$ is obtained as follows

$$U(t) = t^8 - t^7 + t^6 + \frac{2t^{11/2}}{\Gamma(13/2)} - \frac{t^{9/2}}{\Gamma(11/2)} + \frac{t^{7/2}}{\Gamma(9/2)} \qquad (22)$$

In a similar way the equation (14) also gives the following solution after applying the Elzaki transform

$$U(t) = t^{2n+2} - t^{2n+1} + t^{2n} + \frac{2t^{\beta+2}}{\Gamma(\beta+3)} - \frac{t^{\beta+1}}{\Gamma(\beta+2)} + \frac{t^{\beta}}{\Gamma(\beta+1)} \qquad (23)$$

Equations (15), (19) and (23) are same, which shows that the fractional differential equations have found the unique solutions after applying different transforms methods

3. Conclusion

Fractional differential also have the unique solution if the same fractional differential equation solved by different integral transforms methods in a similar manner as in case of classical differential equations.

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