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Analytical Analysis of Two - Dimensional Consolidation of Soil

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Abstract

Consolidation is the gradual reduction in volume of a saturated soil due to drainage of some of the pore water, the process continuing until the excess pore water pressure set up by an increase in total stress has completely dissipated; the most common case is that of one dimensional consolidation. In reality, the Terzaghi's 1-dimensional consolidation theory has been found to be highly conservative and at best only an estimation of the actual consolidation. Accurately predicting consolidation in soil has led to the development of 2-dimensional consolidation solutions. In this paper analytical solution (using the separation of variables method) have been provided for 2-dimensional consolidation equation.

Keywords: Consolidation; Two-dimensional; Pore water pressure; Analytical method, Soil.

1. Introduction

In many real problems of soil mechanics, the conditions are basically two-dimensional as in the case of consolidation of hydraulically deposited soils. Terzaghi in 1925 proposed the first theory to consider the rate of consolidation for saturated cohesive soils. The theory was based on certain assumptions including that the flow of water during consolidation is only in the vertical direction. Some of the assumptions made by Terzaghi are not fully satisfied in actual field problems.

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The results obtained from the use of his theory in solving practical problems are at best approximations. According to [1], one of the major limitations of the Terzaghi's theory is that in the field, the consolidation is usually not one dimensional. Several researchers [2-7] have made advances towards Terzaghi's theory of consolidation into two dimensions. Also, in the work of Razouki and his colleagues [8], design charts for the rate of settlement of embankments on soft soils on the basis of 2D consolidation during construction and post construction periods were developed. They also concluded that the post construction settlement becomes insignificant for relatively thick deposit with permeability in the horizontal direction much higher than that in the vertical direction. In this research, an analytical solution was developed herein to provide acceptable solutions for a two dimensional consolidation problem.

2. One Dimensional Analytical Solution to Consolidation Equation

The solution of the basic differential equation of one-dimensional equation can be obtained using Fourier series [1]. Let us express the hydrostatic excess pressure $\bar{u}$ as;

$$\bar{u} = f_1(z) * f_2(t) \tag{1}$$

Where;

$f_1(z)$ & $f_2(t)$ indicate some function of z and t, respectively.

Substituting the above value of u into the consolidation equation shown in equation (1)

$$C_v\left[f_2(t)\frac{\partial^2}{\partial z^2}[f_1(z)]\right] = f_1(z)\frac{\partial[f_2(t)]}{\partial t}$$

Or

$$\frac{\frac{\partial^2}{\partial z^2}[f_1(z)]}{f_1(z)} = \frac{\frac{\partial}{\partial t}[f_2(t)]}{C_v f_2(t)}$$

The left-hand side of the above equation is a function of z only and the right-hand side is a function of t only. In order words, if the left-hand side is equal to some constant (say, -A2) when t is taken as a variable and the right-hand side is equal to the same constant when z is considered as a variable.

Thus,

$$\frac{\partial^2}{\partial z^2}[f_1(z)] = -A^2 f_1(z)$$

And

$$\frac{\partial^2}{\partial z^2}[f_2(t)] = -A^2 C_v f_2(t)$$

Equation (1) has the solution given by

$$f_1(z) = C_1 cosAz + C_2 \sin Az \qquad (2)$$

where:

$C1$ and $C2$ are constants of integration

e is the base of the hyperbolic or Napierian logarithm.

Substituting the above equations into eqn. (2),

$$\bar{u} = [C_1 cosAz + C_2 \sin Az]C_3 e^{-A^{\Lambda}2C_v t}$$

$$\bar{u} = [C_4 cosAz + C_5 \sin Az]e^{-A^{\Lambda}2C_v t} \qquad (3)$$

Where:

C_4 and C_5 are other constants, such that

$C_5 = C_1 C_3$ and $C_5 = C_2 C_3$

The constants C_4 and C_5 can be determined from the boundary conditions:

(i) t = 0, $\bar{u} = \bar{u}_i$, for any value of z

where $\bar{u}_i$ is initial hydrostatic pressure

(ii) $t = \infty$ $\bar{u} = 0$, for any value of z

(iii) $z = 0$ $\bar{u} = 0$, for any value of t

(iv) $z = H(= 2d)$, $\bar{u} = 0$, for any value of t

For the boundary condition (iii) Equation (3), gives C_4= 0.

Therefore Equation. (3) becomes

$$\bar{u} = C_5 \sin(Az) e^{-A^{\Lambda}2C_v t}$$

For boundary condition iv,

$\bar{u} = 0$ at $z = H$

Therefore,

$C_5 \sin(Az)e^{-A^\wedge 2C_vt} = 0$

The above equation is satisfied if

$AH = n\pi,$

Where:

n is any integer.

The equation can be written in the following form:

$$\overline{u} = B_1 \sin\left(\frac{n\pi}{H}\right) e^{-(pi^\wedge 2/H\Lambda 2)C_vt} + B_2 \sin\left(\frac{2\pi z}{H}\right) e^{-(4pi^\wedge 2/H\Lambda 2)C_vt} + \dots + B_n \sin\left(\frac{n\pi z}{H}\right) e^{-n\Lambda 2*\mathrm{pi}\Lambda 2/H\Lambda 2)C_vt} + \dots..$$

Or

$$\overline{u} = \sum_{n=1}^{n=\infty} B_n \sin\left(\frac{n\pi z}{H}\right) e^{-n\Lambda 2*\mathrm{pi}\Lambda 2/H\Lambda 2)C_vt} \quad (4)$$

Where;

$B_1, B_2, \dots \dots B_n$ are constants.

3. Two Dimensional Analytical Solution to Consolidation Equation

As stated by [1], the governing equation for two dimensional consolidations is given below;

$$m_v\gamma_w \frac{\partial \bar{u}}{\partial t} = k_x \frac{\partial^2 \bar{u}}{\partial x^2} + k_z \frac{\partial^2 \bar{u}}{\partial z^2} \quad (5)$$

Where $k_x = k_z = K$, we have

With boundary conditions;

U(0, z, t) = 0

U(x, 0, t) = 0

U(a, z, t) = 0

U(x, b, t) = 0

And initial condition

$U(x, z, 0) = 100$

Using the separation of variables method,

Let equation (5) be written in the form of

$U(x, z, t) = 0\, w(x, z)\, \tau(t)$

and rearranged, we have

$$\frac{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}}{w} = \frac{\tau'}{\tau}\left(\frac{m_v \gamma_w}{K}\right)$$

For a function of x and z to be identically equal to a function of t for all x, z and t, both sides of this equation must be equal to a constant. For it to decay in time, we should anticipate a negative separation constant.

$$\frac{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}}{w} = \frac{\tau'}{\tau}\left(\frac{m_v \gamma_w}{K}\right) = -\propto^2 \qquad (6)$$

Equation (6) can be separated into space and time

For time we have;

$$\tau' + \tau H \propto^2 = 0 \qquad (7)$$

Where;

$$H = \frac{k}{m_v \gamma_w}$$

For space, we obtain;

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} + \propto^2 w = 0$$

And can be further separated into the product;

$w(x, z) = X(x)\, Z(z)$

And dividing through by XZ, we have;

$$\frac{X''}{X} = \frac{-Z''}{Z} - \propto^2 = \beta^2 \tag{8}$$

Where:

$$\frac{\partial^2 X}{\partial x^2} = X''$$

$$\frac{\partial^2 Z}{\partial z^2} = Z''$$

β^2 is the second separation constant and the sign associated with β^2 remains to be verified.

Thus;

$$X'' - \beta^2 X = 0 \tag{9}$$

$$Z'' + (\propto^2 + \beta^2)Z = 0 \tag{10}$$

From the boundary conditions, we have;

$X(0) = X(a) = 0$

$Z(0) = Z(b) = 0$

The general solution of equation (9) is;

$$X(x) = ASinh\,\beta x + Bcosh\beta x \tag{11}$$

Applying the boundary conditions,

When;

$x = 0, \qquad B = 0$

$x = a, \qquad 0 = A\sinh(\beta . a)$

Implies that

$X(x) = 0 \quad and \quad U(x, z, t) = 0$

Since U=0 does not satisfy the initial time condition, we conclude that the sign of the second separation constant was not chosen correctly. If we replace β^2 with $-\beta^2$, then the separated ordinary differential equation becomes

$\boldsymbol{x}'' + \beta^2 x = 0$

$\boldsymbol{z}'' + (\propto^{\mathbf{2}} - \beta^2)Z = 0$

$X(x) = Asin\beta x + Bcos\beta x$

From applying the boundary conditions, we have,

$B = 0 \quad and \quad \sin\beta a = 0$

$\beta a = m\pi$

$\beta = \frac{m\pi}{a}$

Where; m=1, 2, 3, 4, 5.........

Therefore, equation (11) becomes;

$$X(x) = \ Asin\left(\frac{m\pi}{a}x\right) \quad (12)$$

The general solution of equation (10) is

$Z(z) = \mathrm{C}\sin(YZ) + \ D\cos(YZ)$

From the boundary conditions,

D = 0

Sin (Yb) = 0

Thus;

$Yb = n\pi$

$Y = \frac{n\pi}{b}$

Where; n = 1, 2, 3,.......

Therefore;

$$Z(z) = C\ \sin\left(\frac{n\pi}{b}Z\right) \quad (13)$$

The solution for equation (7) is

$$\tau(t) = e^{-H\propto^2 t} \qquad (14)$$

Let;

$\propto^2 = \beta^2 + y^2$

Applying this to equation (10), we have that;

$\propto^2 = \beta^2 + y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

The three solutions of equation (12), (13), (14) can now be combined to yield

$U(x,z,t) = X(x)Z(z)\,\tau(t) = AC\ \sin\left(\frac{m\pi}{a}x\right)\ \sin\left(\frac{n\pi}{b}Z\right) e^{-H\propto^2 t}$

$U(x,z,t) = Esin\left(\frac{m\pi}{a}x\right).\sin\left(\frac{n\pi}{b}z\right) e^{-H\left[\left(\frac{\cup\pi}{a}\right)^2 + \left(\frac{na}{b}\right)^2\right]t}$

Where;

$E = AC$

To satisfy the time condition a linear combination for all positive integers of m and n is required.

$$U(x,z,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn} sin\left(\frac{m\pi}{a}x\right).\sin\left(\frac{n\pi}{b}z\right) e^{-H\left[\left(\frac{\cup\pi}{a}\right)^2 + \left(\frac{na}{b}\right)^2\right]t} \qquad (15)$$

The initial conditions require that

$U(x,z,0) = 100 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn} sin\left(\frac{m\pi}{a}x\right).\sin\left(\frac{n\pi}{b}z\right)$

Using Fourier sine orthogonality relations to evaluate the coefficient E_{mn}, we multiply by $sin\left(\frac{\overline{m}\pi}{a}x\right)$and integrate from 0 to a to obtain

$\int_0^a 100 sin\left(\frac{\overline{m}\pi}{a}x\right)dx = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn} sin\left(\frac{nxz}{b}\right)\int_0^{\mathrm{a}} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{\overline{m}\pi}{a}x\right)\mathrm{dx}$

From orthogonality relation

$$\int_0^a sin\left(\frac{\overline{m}\pi}{a}x\right)\sin\left(\frac{\overline{m}\pi}{a}x\right)dx = \begin{cases} 0, m \neq \overline{m} \\ \frac{a}{2}, m = \overline{m} \end{cases}$$

Therefore;

$$\int_0^a 100sin\left(\frac{\overline{m}\pi}{a}x\right)dx = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn}sin\left(\frac{nxz}{b}\right).\frac{a}{2}$$

But;

$$\sum_{n=1}^{\infty}\frac{a}{2} = \frac{a}{2}$$

Hence, we have;

$$\int_0^a 100sin\left(\frac{\overline{m}\pi}{a}x\right)dx = \frac{a}{2}\sum_{m=1}^{\infty} E_{mn}sin\left(\frac{n\pi z}{b}\right)$$

Multiplying by $sin\left(\frac{n\pi z}{b}\right)$ and integrating from 0 to b.

$$\int_0^b\int_0^a 100sin\left(\frac{\overline{m}\pi}{a}x\right)sin\left(\frac{n\pi z}{b}\right)dx\,dz = \int_0^b\frac{a}{2}\sum_{n=1}^{\infty} E_{mn}sin\left(\frac{n\pi z}{b}\right)sin\left(\frac{n\pi z}{b}\right)dz$$

Applying the orthogonality relation

$$\int_0^b\int_0^a 100sin\left(\frac{\overline{m}\pi}{a}x\right)sin\left(\frac{n\pi z}{b}\right)dx\,dz = \frac{a}{2}\sum_{m=1}^{\infty} E_{mn}\frac{b}{2}$$

Where,

$$\sum_{n=1}^{\infty}\frac{b}{2} = \frac{b}{2}$$

Hence;

$$\int_0^b\int_0^a 100sin\left(\frac{\overline{m}\pi}{a}x\right)sin\left(\frac{n\pi z}{b}\right)dx\,dz = \frac{a}{2}.\frac{b}{2}E_{mn} = \frac{ab}{4}E_{mn}$$

Where;

$$\overline{m} = m$$

$$\bar{n} = n$$

Therefore,

$$\int_0^b \int_0^a 100 sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi z}{b}\right) dx\, dz = \frac{ab}{4} E_{mn}$$

$$E_{mn} = \frac{4}{ab}\int_0^b \int_0^a 100 sin\left(\frac{m\pi}{a}x\right) sin\left(\frac{n\pi z}{b}\right) dx\, dz \qquad (16)$$

Equation (15) is the exact solution to equation (5) with E_{mn} computed from equation (16).

4. Case Study

The soil for study is that of a highly compressible clay layer, 2m thick and is subjected to a vertical pressure of 100kPa that is maintained constant with time. Drainage is allowed from both the vertical and horizontal direction. From laboratory analysis carried out on soil sample obtained, the following data were obtained.

Table 1: Soil Properties

S/N	Parameters	Value
1	Moisture content	88.0
2	Specific gravity	2.59
3	Liquid limit	140
4	Plastic limit	37
5	Plasticity Index	103
6	Consolidation modulus, E_{oed} (MN/m^2)	0.2
7	Coefficient of volume compressibility, m_v (MN/m^2)	5.7E-010
8	Coefficient of consolidation Pressure, Cv, (m/S^2)	3.4E-09
9	Compression index Cc	9.9366
10	Coefficient of permeability in the vertical direction, $k_{z.,}$ (m/S^2)	6.8E-11

The natural moisture content test result is 88%, with specific gravity 2.59, the Atterberg's limit of 140%, plastic limit of 37% and plasticity index of 103%. The soil is highly plastic. The result for the pore water pressure using equation (15) at t = 10, 100, 600 and 1000 days for the two dimensional consolidation is presented in Figure 1.

The result of the predicted pore water pressure using the developed analytical method showed a decrease in pore water pressure at various depth with time.

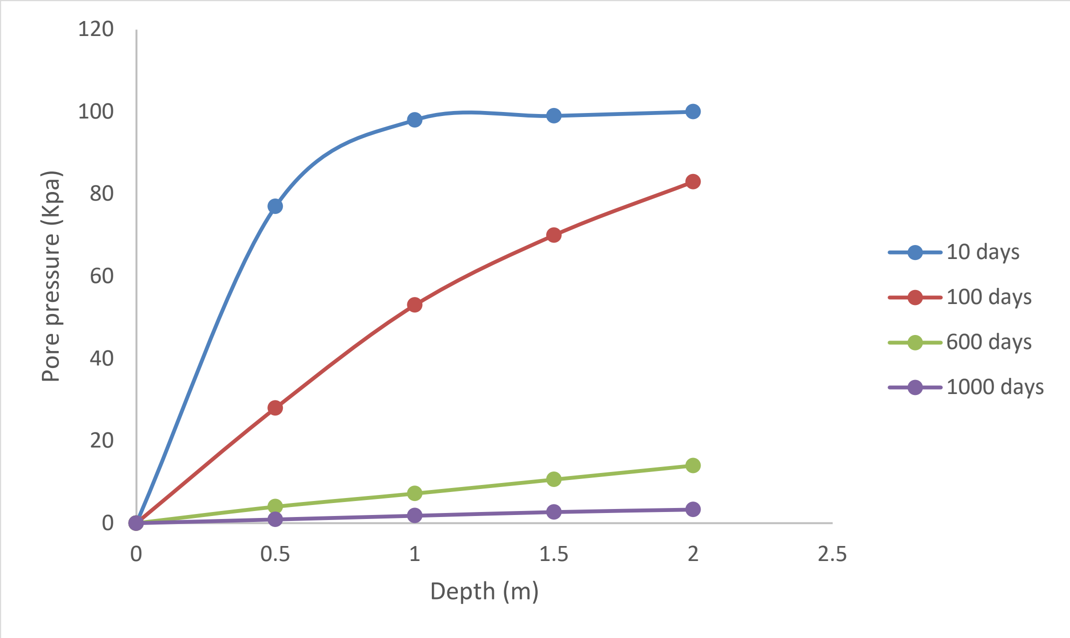

Figure 1: Pore water pressure plot

5. Conclusion

The one dimensional consolidation problem developed by Terzaghi has been expanded to a two dimensional to help simulate field condition. The validation of the equation, was carried out using consolidation parameters obtained from laboratory test carried out on a cohesive soil. The initial pore pressure of 100kPa can be seen to be reducing as the time increases. The equation can be used for two dimensional consolidation problem.

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