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The Relationship of Fractional Laplace Transform with Fractional Fourier, Mellin and Sumudu Transforms

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Abstract

We have developed in this research paper, some of the fundamental relationship of fractional Laplace transform with fractional Fourier, fractional Mellin and fractional Sumudu transforms. These results are expressed mathematically, and such relationships should be very useful in applications to signal processing and optics.

Keywords: Fractional Laplace Transform; Fractional Mellin Transform; Fractional Fourier Transform; Fractional Sumudu transform.

1. Introduction

Fractional integral transforms provide a well-established and valuable method for solving problems in many areas of applied mathematics, physics like optics, signal processing, quantum mechanics. Since the introduction of fractional Fourier transform by Namias in 1980 [1], the applied mathematicians, physicists are paying their attention not only on fractional Fourier transforms but also working on many other transforms like fractional Hilbert transform fractional Mellin transform fractional Laplace transform and fractional Sumudu transform. In recent times the fractional integral transforms have become a very important tool and are playing a key role in various branches of applied mathematics and physics. Fractional Laplace transform was introduced by many researchers in their research articles in different ways, in 2003 it was first defined by A. Torre as a special case of canonical transform with characteristic matrix and its relation to canonical transform, and parabolic differential equations are discussed in [2], the properties of fractional Laplace transform are also developed [3].

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In 2009 Guy Jumarie developed a new form of fractional Laplace transform with the Mittage Leffler function for the entire class of functions that are fractional differentiable [4]. In 2010 in K. K. Sharma defined fractional Laplace Transform as a special case of linear canonical transform with representative matrix and used it in problems [5]. The convolution structure for the two versions of fractional Laplace transform was developed [6]. Various of fractional Laplace transform properties are discussed which are useful in application to differential and integral equations or problems in non-extensive statistical mechanics [7]. In this paper we have established the relationship of fractional Laplace transform with other transforms.

The fractional Laplace transform is defined in [5]

$$\mathcal{L}^{\alpha}\{\phi(t)\} = \bar{\phi}^{\alpha}(s) = \int_{-\infty}^{\infty} \phi(t) k_{\alpha}(t,u)\, dt \qquad (1)$$

Where $k_{\alpha}(t,u)$ is called kernel and it is defined as

$$k_{\alpha}(t,u) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, e^{i\frac{\cot\alpha}{2}(t^2 - s^2 + 2ist\sec\alpha)} \qquad (2)$$

This reduces into classical Laplace transform if we put $\alpha = \frac{\pi}{2}$. The inversion formula for fractional Laplace transform can be obtained by replacing α by $-\alpha$ in equations (1) and (2) Properties of kernel function of Laplace transform stated in [3] by Gudadhe are given as follows

$$\begin{cases} k_{\alpha}(t,u) = k_{\alpha}(u,t) \\ k_{-\alpha}(t,u) = {k_{\alpha}}^{*}(t,u) \\ k_{\alpha}(-t,u) = k_{\alpha}(t,-u) \\ \int_{-\infty}^{\infty} k_{\alpha}(t,u) k_{\beta}(u,z) du = k_{\alpha+\beta}(t,z) \\ \int_{-\infty}^{\infty} k_{\alpha}(t,u)\, {k_{\alpha}}^{*}(t,u) dt = \delta(u - u') \end{cases} \qquad (3)$$

The kernel which is defined in eq (3) has the similar expressions as Almeida defined [8].

2. Results and Discussions

In this section we are presenting some important relationship of fractional Laplace transform with other transform like fractional Fourier transform, fractional Mellin transform and fractional Sumudu transform which can play a significant role in signal processing and other fields of applied mathematics

2.1. The relationship between fractional Laplace transform with fractional Fourier transform

Since fractional Fourier transform of $\phi(t)$ is defined as [8]

$$\mathcal{F}^{\alpha}\{\phi(t)\} = \bar{\phi}^{\alpha}(\omega)$$
$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\phi(t)e^{\frac{i\cot\alpha}{2}(t^2+\omega^2)-(i\omega)t\csc\alpha}dt, \text{when } \alpha \text{ is not a multiple of } \pi \quad (5)$$

The fractional Laplace transform is defined as after combining equations (1) and (2)

$$\mathcal{L}^{\alpha}\{\phi(t)\} = \bar{\phi}^{\alpha}(s)$$
$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\phi(t)e^{i\frac{\cot\alpha}{2}(t^2-s^2+2ist\sec\alpha)}\,dt, \text{when } \alpha \text{ is not a multiple of } \pi \quad (6)$$

since s is complex therefore substituting $s = \sigma + i\omega$ in in equation (6)

we get

$$\mathcal{L}^{\alpha}\{\phi(t)\} = \bar{\phi}^{\alpha}(\sigma + i\omega) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\phi(t)e^{\frac{i\cot\alpha}{2}(t^2-(\sigma+i\omega)^2)-(\sigma+i\omega)t\csc\alpha}dt$$
$$= \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\phi(t)e^{\frac{i\cot\alpha}{2}(t^2-\sigma^2+\omega^2-2i\sigma\omega)-(\sigma+i\omega)t\csc\alpha}dt$$

$$= e^{-\frac{i\cot\alpha}{2}(\sigma^2+2i\sigma\omega)}\sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\phi(t)e^{-\sigma t\csc\alpha}e^{\frac{i\cot\alpha}{2}(t^2+\omega^2)-(i\omega)t\csc\alpha}dt$$

$$= e^{-\frac{i\cot\alpha}{2}(\sigma^2+2i\sigma\omega)}\left[\sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}(\phi(t)e^{-\sigma t\csc\alpha})e^{\frac{i\cot\alpha}{2}(t^2+\omega^2)-(i\omega)t\csc\alpha}dt\right] \quad (7)$$

From (5) and (7) we get the result

$$\mathcal{L}^{\alpha}\{\phi(t)\} = e^{-\frac{i\cot\alpha}{2}(\sigma^2+2i\sigma\omega)}\mathcal{F}^{\alpha}\{\phi(t)e^{-\sigma t\csc\alpha}\} \quad (8)$$

Eq (8) shows the relationship of fractional Laplace transform with fractional Fourier transform and this relation will reduces to classical relation of Laplace transform and Fourier transform if we put $\alpha = \frac{\pi}{2}$.

2.2. Relation between Fractional Laplace Transform and Fractional Mellin Transform

In order to develop a relation of FRLT with FRMT let us consider the change of variable which is defined by

$$e^{-t} = y$$

Replacing t by e^{-t} in equation (6) we get

$$\mathcal{L}^{\alpha}\{\phi(e^{-t})\}(s) = \sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_{-\infty}^{\infty} \phi(e^{-t}) e^{\frac{i\cot\alpha}{2}(t^2-s^2)-st\csc\alpha} dt \qquad (9)$$

Substituting $e^{-t} = y \implies dt = -\frac{dy}{y}$

When $t \to -\infty$ then $y \to \infty$, when $t \to \infty$ then $y \to 0$ in equation (6) we get

$$= -\sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_{\infty}^{0} \phi(y) e^{\frac{i\cot\alpha}{2}(\ln y^2-s^2)+s\,csc\,\alpha\ln y} \frac{dy}{y}$$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_{0}^{\infty} \phi(y) e^{\frac{i\cot\alpha}{2}(\ln y^2-s^2)+\ln y^{s\csc\alpha}} \frac{dy}{y}$$

$$= \sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_{0}^{\infty} \phi(y) e^{\frac{i\cot\alpha}{2}(\ln y^2-s^2)} y^{s\csc\alpha-1} dy \qquad (10)$$

In [3] fractional Mellin transform is defined as

$$\mathcal{M}^{\alpha}\{\phi(y)\} = \sqrt{\frac{1-i\cot\alpha}{2\pi}} \int_{0}^{\infty} \phi(y) e^{\frac{i\cot\alpha}{2}(\ln y^2-s^2)} y^{s\csc\alpha-1} dy \qquad (11)$$

From (10) and (11) we get the relation between fractional Laplace transform and fractional Mellin transform

$\mathcal{L}^{\alpha}\{\phi(e^{-t})\} = \mathcal{M}^{\alpha}\{\phi(y)\}$

This reduces to classical relation between Laplace transform Mellin transform for $\alpha = \frac{\pi}{2}$.

2.3. Relation between Fractional Laplace Transform and Fractional Sumudu Transform

In [10] the duality relation between two sided Laplace transform and two sided Sumudu transform is given by

$$G(u) = \frac{1}{u} F(s)\Big|_{s=\frac{1}{u}} \qquad (12)$$

Or

$$F(s) = \frac{1}{s} G(u)\Big|_{u=\frac{1}{s}} \qquad (13)$$

Where $G(u) = S\{f(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$ are the Sumudu and Laplace transform respectively

The generalized fractional Sumudu transform can be established by using the duality relation between Laplace transform and Sumudu transform as defined in a similar way as defined in equations (12) and (13)[11]

Let $\phi(t)$ be of exponential order and $\mathcal{L}^{\alpha}\{\phi(t)\} = \bar{\phi}^{\alpha}(s)$ and $S^{\alpha}\{\phi(t)\} = G^{\alpha}(u)$ then

$$G^{\alpha}(u) = \frac{1}{u}\bar{\phi}^{\alpha}\left(\frac{1}{u}\right) = \frac{1}{u}\sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\phi(t)e^{\frac{i\cot\alpha}{2}\left(t^2-(\frac{1}{u})^2\right)-\frac{t}{u}\csc\alpha}dt \qquad (14)$$

and

$$G^{\alpha}(u) = S^{\alpha}\{\phi(t)\} = \frac{1}{u}\sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\phi(t)e^{\frac{i\cot\alpha}{2}\left(t^2-(\frac{1}{u})^2\right)-\frac{t}{u}\csc\alpha}dt \quad (15)$$

Substituting $u = \frac{1}{s}$ we get

$$G^{\alpha}\left(\frac{1}{s}\right) = S^{\alpha}\{\phi(t)\} = s\sqrt{\frac{1 - i\cot\alpha}{2\pi}}\int_{-\infty}^{\infty}\phi(t)e^{\frac{i\cot\alpha}{2}(t^2-(s)^2)-st\csc\alpha}dt \rightarrow (16)$$

From (16) and (1) we get the relation of fractional Laplace transform with fractional Sumudu transform

$$\frac{1}{s}G^{\alpha}\left(\frac{1}{s}\right) = \bar{\phi}^{\alpha}(s) \qquad (17)$$

3. Conclusion

We have established mathematically the relationship of the fractional Laplace transform with fractional Mellin transform and Fractional Fourier transform which will play a significant role in signal processing, optics and other field of applied mathematics, physics and engineering

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