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The Relationship of Generalized Fractional Hilbert Transform with Fractional Mellin and Fractional Laplace Transforms

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Abstract

We have developed in this research paper, some of the fundamental relationship between generalized fractional Hilbert transform with fractional Mellin transform, fractional Laplace transform, fractional inverse Laplace transform.. The results are mathematically expressed. These results, however, need modelling and simulation with any specialized signal processing data.

Keywords: Fractional Hilbert Transform; Fractional Mellin Transform; Fractional Laplace Transform; Fractional Fourier transform.

1. Introduction

Fractional Hilbert transform(FRHT) is introduced by Lohmann and his colleagues [1]. They proved in their paper that the FRHT is the generalization of the Hilbert transform(HT). Their generalization is based on modifying the spatial filters and fractional Fourier plane for filters.

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In continuation of the work [1], Davis and his colleagues [2] employed FRFT for edge detection and proved that as the fractional order is varied then different qualities of edge enhancement are obtained. In [3] discrete counterpart of the FRHT is discussed. Zayed [4] took the analytical signal (AS) formalism associated with the standard Fourier transform (FT) and provided a counterpart of it for the fractional Fourier transform (FRFT). The communication applications is discussed by using FRHT in [5] Fractional Hilbert transform is a special case of linear canonical transform (LCT) which has diverse applications such as in an image enhancement or compression by using the angle of rotation in the complex plane (t,w) of fractional Fourier transform (FRFT) on optical systems, edge deduction for propagation and delay times , beam flowing and indeed in signal processing [6-13]. Needless to mention, the integral transforms have significant applications in both Physics and Applied Mathematics. Akilahmad Sheikh and Alka Gudadhe [14,15] worked on relationships on generalized fractional Hilbert transform with some classical transforms and developed analytic theorems. The fractional Hilbert transform is a generalization of Hilbert transform and so is the case of fractional Fourier tansform (FRFT). The scale invariance property of fractional Mellin transform (FMT) is a very important tool to constructing two dimensional real images [16]. Analysis of Hilbert transforms with fractional Fourier transform (FRFT) [17-18] resulted into many fascinating results. The use of fractional Hilbert transforms and its extensions lead to analytical behavior of signal constructions. [9]. This is how amplitude modulation (AM,) frequency modulation (FM) or phase modulation(PM) can be exploited in Weiner filter for controlling the changes in phase signals. These fractional phase changes are the manifestations of the Fractional Hilbert transform and of the Morlet and Harlet wavelets for two super posed wave form. The same behavior is witnessed with Fourier transform [4]. The Fractional Laplace transform is also the genialized case of Laplace transform. Heaviside step function is a step forward to deal with Laplace transform because it deals in the denominator with a function with fractional exponent. Laplace and its corresponding fractional Laplace transform deals with Wigner distribution(WD), Wiener space(WS) the ambiguity fraction, the short time Fourier transform(SSTFT), speech processing, radar, image rotation, Confocal microscopy, etc. The aperiodic stable chaos with Lyapunov experiments in real time signals can be studied with fractional Laplace transform [19-20].

2. Results and Discussions

2.1. The relationship between generalized fractional Hilbert transform with fractional Mellin transform

The relation between Fractional Mellin transform and fractional Fourier is defined in [16] as

$$FRMT[f(x)] = FRFT[f(e^t)]$$

$$= \sqrt{\frac{1 - icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(e^t)\, e^{-iutcosec\alpha} e^{\frac{i}{2}(t^2+u^2)cot\alpha} dt \qquad (1)$$

FRFT is the fractional Fourier transform and its definition and its applications are disused in [21-23].

Where FRMT is fractional Mellin transform and defined as

$$M^{\alpha}(u) = FRMT[f(t)] = \sqrt{\frac{1 - icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iutcosec\alpha} e^{\frac{i}{2}(t^2+u^2)cot\alpha} dt$$

The generalize fractional Hilbert transform is defined in [14] as

$$H^{\alpha}[f(x)](t) = \frac{e^{-i\frac{cot\alpha}{2}t^2}}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{t-x} e^{i\frac{cot\alpha}{2}x^2} dx \qquad (2)$$

$$H^{\alpha}\left\{e^{\frac{i}{2}(-u^2+y^2)cot\alpha} M^{\alpha}[f(t)](u)\right\}(y) = \frac{e^{-i\frac{cot\alpha}{2}y^2}}{\pi} \int_{-\infty}^{\infty} e^{(-u^2+y^2)cot\alpha} \frac{M^{\alpha}[f(t)]}{y-u} e^{i\frac{cot\alpha}{2}u^2} du$$

$$= \frac{e^{-i\frac{cot\alpha}{2}y^2}}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i\frac{cot\alpha}{2}u^2}}{y-u} \left\{ \sqrt{\frac{1 - icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(e^t)\, e^{-iutcosec\alpha} e^{\frac{i}{2}(t^2+u^2)cot\alpha} dt \right\} du$$

$$= \frac{e^{i\frac{cot\alpha}{2}y^2}}{\pi} \sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\frac{cot\alpha}{2}t^2}}{y-u} \left\{\int_{-\infty}^{\infty} f(e^t)\, e^{-iutcosec\alpha} dt\right\} du$$

By changing the order of integration

$$= e^{i\frac{cot\alpha}{2}y^2} \sqrt{\frac{1 - icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(e^t) e^{i\frac{cot\alpha}{2}t^2} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-iutcosec\alpha}}{y-u} du \right\} dt \qquad (3)$$

From [18] we have

$$\int_{-\infty}^{\infty} \frac{e^{iut}}{x-t} dt = -i\, sgn(u) e^{iux} \qquad (4)$$

equation (3) can be written as

$$= e^{i\frac{cot\alpha}{2}y^2} \sqrt{\frac{1 - icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(e^t)\, e^{i\frac{cot\alpha}{2}t^2} (i) sgn\left(\frac{t}{sin\alpha}\right) e^{-iytcosec\alpha} dt$$

$$= i\sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(e^t)\, sgn\left(\frac{t}{sin\alpha}\right) e^{-iytcosec\alpha} e^{\frac{i}{2}(t^2+y^2)cot\alpha} dt$$

$$H^{\alpha}\{M^{\alpha}[f(t)]\}(u) = iM^{\alpha}\left[Sgn\left(\frac{t}{sin\alpha}\right) f(t)\right] \qquad (5)$$

Equation (4) is the relation between generalized fractional Hilbert transform and Fractional Mellin transform.

2.2. The relationship between generalized fractional inverse Hilbert transform with fractional Mellin transform

Let $$g(u) = \int_{-\infty}^{\infty} f(t)\, e^{-iutcosec\alpha} e^{\frac{i}{2}t^2 cot\alpha} dt \qquad (5)$$

$$e^{i\frac{cot\alpha}{2}u^2} \sqrt{\frac{1-icot\alpha}{2\pi}}\, g(u) = \sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-iutcosec\alpha} e^{\frac{i}{2}(t^2+u^2)cot\alpha} dt$$

$$e^{i\frac{cot\alpha}{2}u^2} \sqrt{\frac{1-icot\alpha}{2\pi}}\, g(u) = M^{\alpha}[f(t)](u) \qquad (6)$$

$$H^{-\alpha}\{M^{\alpha}[f(t)](u)\}(y) = -\frac{e^{i\frac{cot\alpha}{2}y^2}}{\pi} \int_{-\infty}^{\infty} \frac{M^{\alpha}[f(t)](u)}{y-u} e^{-i\frac{cot\alpha}{2}u^2} du$$

$$= \frac{-e^{i\frac{cot\alpha}{2}y^2}}{\pi} \int_{-\infty}^{\infty} \frac{1}{y-u} e^{-i\frac{cot\alpha}{2}u^2} \sqrt{\frac{1-icot\alpha}{2\pi}}\, e^{i\frac{cot\alpha}{2}u^2} g(u) du$$

$$= \frac{-e^{i\frac{cot\alpha}{2}y^2}}{\pi} \sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty} \frac{1}{y-u} \left\{ \int_{-\infty}^{\infty} f(t)\, e^{-iutcosec\alpha} e^{\frac{i}{2}t^2 cot\alpha} dt \right\} du$$

$$by\ changing\ the\ order\ of\ integration$$

$$H^{-\alpha}\{M^{\alpha}[f(t)](u)\}(y) = \frac{-e^{i\frac{cot\alpha}{2}y^2}}{\pi} \sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{\frac{i}{2}t^2 cot\alpha} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{y-u} e^{-iutcosec\alpha} du \right\} dt \qquad (7)$$

We have from [18] $$\int_{-\infty}^{\infty} \frac{e^{iut}}{x-t} dt = -i\, sgn(u) e^{iux} \qquad (8)$$

Equation (7) can be written as

$$= \frac{-e^{i\frac{cot\alpha}{2}y^2}}{\pi} \sqrt{\frac{1-icot\alpha}{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{\frac{i}{2}t^2 cot\alpha} .\, i.\, sgn\left(\frac{t}{sin\alpha}\right) e^{-iytcosec\alpha} dt$$

$$H^{-\alpha}\{M^{\alpha}[f(t)](u)\}(y) = -iM^{\alpha}\left[Sgn\left(\frac{t}{sin\alpha}\right) f(t)\right](y) \qquad (9)$$

Equation (9) is the relation between generalized fractional inverse Hilbert transform and Fractional Mellin transform.

2.3 Relationship between generalized fractional invers Hilbert transform with fractional Laplace transform

Fractional Laplace transform is defined in [24] as

$$L^{\alpha}[f(t)](u) = e^{-i\frac{cot\alpha}{2}u^2}\sqrt{\frac{1}{2\pi sin\alpha}}\int_{-\infty}^{\infty} f(t)\, e^{-utcosec\alpha} e^{\frac{i}{2}t^2 cot\alpha} dt \qquad (10)$$

Using equation (5)

$$e^{-i\frac{cot\alpha}{2}u^2}\sqrt{\frac{1}{2\pi sin\alpha}}\, g(u) = e^{-i\frac{cot\alpha}{2}u^2}\sqrt{\frac{1}{2\pi sin\alpha}}\int_{-\infty}^{\infty} f(t)\, e^{-utcosec\alpha} e^{\frac{i}{2}t^2 cot\alpha} dt$$

$$e^{-i\frac{cot\alpha}{2}u^2}\sqrt{\frac{1}{2\pi sin\alpha}}\, g(u) = L^{\alpha}[f(t)](u) \qquad (11)$$

Now consider

$$H^{-\alpha}\left\{e^{icot\alpha(u^2-y^2)}L^{\alpha}[F(t)](u)\right\}(y) = -\frac{e^{i\frac{cot\alpha}{2}y^2}}{\pi}\int_{-\infty}^{\infty} e^{icot\alpha(u^2-y^2)}\frac{L^{\alpha}[F(t)](u)}{y-u} e^{-i\frac{cot\alpha}{2}u^2} du$$

$$= -\frac{e^{-i\frac{cot\alpha}{2}y^2}}{\pi}\int_{-\infty}^{\infty}\frac{1}{y-u} e^{i\frac{cot\alpha}{2}u^2} e^{-i\frac{cot\alpha}{2}u^2}\sqrt{\frac{1}{2\pi sin\alpha}}\, g(u) du$$

$$= -\frac{e^{-i\frac{cot\alpha}{2}y^2}}{\pi}\sqrt{\frac{1}{2\pi sin\alpha}}\int_{-\infty}^{\infty}\frac{1}{y-u} g(u) du$$

$$= -\frac{e^{-i\frac{cot\alpha}{2}y^2}}{\pi}\sqrt{\frac{1}{2\pi sin\alpha}}\int_{-\infty}^{\infty}\frac{1}{y-u}\left\{\int_{-\infty}^{\infty} f(t)\, e^{-utcosec\alpha} e^{\frac{i}{2}t^2 cot\alpha} dt\right\} du \qquad (12)$$

Interchanging the integration order

$$= -e^{-i\frac{cot\alpha}{2}y^2}\sqrt{\frac{1}{2\pi sin\alpha}}\int_{-\infty}^{\infty} f(t)\, e^{\frac{i}{2}t^2 cot\alpha}\left\{\int_{-\infty}^{\infty}\frac{1}{y-u} e^{-utcosec\alpha} du\right\} dt \qquad (13)$$

$$= -e^{-i\frac{cot\alpha}{2}y^2}\sqrt{\frac{1}{2\pi sin\alpha}}\int_{-\infty}^{\infty} f(t)\, e^{\frac{i}{2}t^2 cot\alpha} sgn\left(\frac{t}{sin\alpha}\right) e^{-ytcosec\alpha} dt \qquad (14)$$

$$H^{-\alpha}\left\{L^{\alpha}\left[F(t)e^{icot\alpha u^2}\right](u)\right\}(y) = -L^{\alpha}\left[\int_{-\infty}^{\infty} f(t)\, sgn\left(\frac{t}{sin\alpha}\right)\right](y) \quad (15)$$

Equation (15) is the relation between generalized fractional Hilbert transform and Fractional Laplace transform.

2.4 The relationship between generalized fractional Hilbert transform with fractional inverse Laplace transforms

The fraction inverse Laplace transform is defined in [24]

$$f(t)=\sqrt{\frac{1+icot\alpha}{2\pi}}\,e^{\frac{i}{2}t^2cot\alpha}\int_{\alpha-\infty}^{\alpha+\infty}F\,(usin\alpha)e^{-\frac{i}{4}t^2sin2\alpha}e^{ut}du$$

$$H^{\alpha}\{e^{-it^2cot\alpha}L^{-\alpha}[f(u)](t)\}(y)=\frac{e^{-\frac{icot\alpha}{2}y^2}}{\pi}\int_{-\infty}^{\infty}e^{it^2cot\alpha}\,\frac{L^{-\alpha}[f(u)](t)}{y-t}\,e^{\frac{i}{2}t^2cot\alpha}dt$$

$$=\frac{e^{-\frac{icot\alpha}{2}y^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-it^2cot\alpha}}{y-t}e^{\frac{i}{2}t^2cot\alpha}\left\{\sqrt{\frac{1+icot\alpha}{2\pi}}\,e^{\frac{i}{2}t^2cot\alpha}\int_{\alpha-\infty}^{\alpha+\infty}F_{L.\alpha}\,(usin\alpha)e^{-\frac{iu^2}{4}sin2\alpha}e^{ut}du\right\}dt$$

$$=e^{-\frac{icot\alpha}{2}y^2}\sqrt{\frac{1+icot\alpha}{2\pi}}\int_{\alpha-\infty}^{\alpha+\infty}F_{L.\alpha}\,(usin\alpha)e^{-\frac{i}{4}t^2sin2\alpha}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ut}}{y-t}dt\right\}du$$

$$=e^{-\frac{icot\alpha}{2}y^2}\sqrt{\frac{1+icot\alpha}{2\pi}}\int_{\alpha-\infty}^{\alpha+\infty}F_{L.\alpha}\,(usin\alpha)e^{-\frac{i}{4}t^2sin2\alpha}.sgn(u)e^{ut}du$$

$$H^{\alpha}\{e^{-it^2cot\alpha}L^{-\alpha}[f(u)](t)\}(y)=L^{-\alpha}[f(u)](y)$$

On the relationship between generalized inverse fractional Hilbert transform with fractional inverse Laplace transform

$$H^{-\alpha}\{e^{it^2cot\alpha}L^{-\alpha}[f(u)](t)\}(y)=\frac{-e^{\frac{icot\alpha}{2}y^2}}{\pi}\int_{-\infty}^{\infty}e^{it^2cot\alpha}\,\frac{L^{-\alpha}[f(u)](t)}{y-t}\,e^{\frac{-i}{2}t^2cot\alpha}dt$$

$$=\frac{-e^{\frac{icot\alpha}{2}y^2}}{\pi}\int_{-\infty}^{\infty}\frac{e^{it^2cot\alpha}}{y-t}e^{-\frac{i}{2}t^2cot\alpha}\left\{\sqrt{\frac{1+icot\alpha}{2\pi}}\,e^{\frac{i}{2}t^2cot\alpha}\int_{\alpha-\infty}^{\alpha+\infty}F_{L.\alpha}\,(usin\alpha)e^{-\frac{iu^2}{4}sin2\alpha}e^{ut}du\right\}dt$$

$$=-e^{\frac{icot\alpha}{2}y^2}\sqrt{\frac{1+icot\alpha}{2\pi}}\int_{\alpha-\infty}^{\alpha+\infty}F_{L.\alpha}\,(usin\alpha)e^{-\frac{i}{4}t^2sin2\alpha}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{ut}}{y-t}dt\right\}du$$

$$=-e^{\frac{icot\alpha}{2}y^2}\sqrt{\frac{1+icot\alpha}{2\pi}}\int_{\alpha-\infty}^{\alpha+\infty}F_{L.\alpha}\,(usin\alpha)e^{-\frac{i}{4}t^2sin2\alpha}.sgn(u)e^{ut}du$$

$$H^{\alpha}\{e^{-it^2cot\alpha}L^{-\alpha}[f(u)](t)\}(y)=-L^{-\alpha}[f(u)](y)$$

3. Conclusion

We have established mathematically the relationship of the fractional Hilbert transform with fractional Mellin transform and Fractional Laplace transform., which will play a significant role in signal processing and other field of applied mathematics, engineering and physics.

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