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A Static Solution to Einstein's Field Equations for a Spherical Distribution of Electrically Counterpoised Dust with a Set of New Boundary Conditions

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Abstract

A static spherically symmetric solution to Einstein's field equations has been found with the standard boundary conditions for a spherical distribution of electrically counterpoised dust in Sashinka Wimaladharma and Nalin de Silva. However the solution obtained there was not Lorentzian at infinity. To overcome this problem, a set of new boundary conditions has been introduced along with different coordinates for different regions of the matter distribution.

Keywords: Einstein's field equations; electrically counterpoised dust.

1. Introduction

Sashinka wimaladharma and Nalin de silva [1] considered a spherically symmetric distribution of electrically counterpoised dust and solved Einstein-Maxwell field equations using the standard boundary conditions which says that metric coefficients and their partial derivatives are continuous on the boundary of the sphere. Also there the metrics had the same form for the two regions inside and outside the sphere with the same coordinates (t, r) both inside and outside the sphere.

Also it has been also found that the metric in the exterior region of the sphere is not tending to Lorentzian form at infinity.

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In order to obtain a Lorentzian metric at infinity for a spherically symmetric distribution of matter, it is assumed that the time and radial coordinates are different for the two regions, inside and outside the sphere and a new set of boundary conditions has been introduced to solve the Einstein-Maxwell equations and equations for a distribution of electrically counterpoised dust (ECD).

2. Material and the Method

Closely following the form of the metric obtained in Sashinka wimaladharma and Nalin de Silva [1], a metric for exterior region of the sphere has taken to be of the form

$$ds^2 = \frac{1}{\left(A_1+\frac{A_2}{R}\right)^2} c^2 dT^2 - \left(A_1 + \frac{A_2}{R}\right)^2 (dR^2 + R^2 d\Omega^2) \quad .$$

The density of a sphere of electrically counterpoised dust is taken to be constant and equal to $\frac{1}{4\pi l^2}$ with $4\pi\rho l^2 = 1$ as in Sashinka wimaladharma and Nalin de Silva [1] .

Now we introduce two sets of coordinates (t, r) and Ω and (T, R) and $\overline{\Omega}$ for the metrics inside and outside of a sphere of constant density, respectively.

Then the metrics for the two regions inside and outside of a sphere of constant density $\rho = \frac{1}{4\pi l^2}$ can be written in the form

$$ds^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2} c^2 dt^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2 (dr^2 + r^2 d\Omega^2) \qquad 0 \le r \le a$$

$$ds^2 = \frac{1}{\left(A_1+\frac{A_2}{R}\right)^2} c^2 dT^2 - \left(A_1 + \frac{A_2}{R}\right)^2 \left(dR^2 + R^2 d\overline{\Omega}^2\right) \qquad A < R \quad . \tag{1}$$

where $d\Omega^2 = d\Theta^2 + \sin^2\Theta\ d\Phi^2$ and $d\overline{\Omega}^2 = d\overline{\Theta}^2 + \sin^2\overline{\Theta}\ d\overline{\Phi}^2$, $\theta\left(\frac{R}{l}\right)$ is the Emden function satisfying the Emden equation [2] with $n = 3$.

The metric for the inside of the sphere is taken to be of the same form

$ds^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2} c^2 dt^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2 (dr^2 + r^2 d\Omega^2)$ which has been found to be the metric inside the sphere in Sashinka wimaladharma and Nalin de Silva [1]., as there are no differences between the coordinates that were used in Sashinka wimaladharma and Nalin de Silva [1], and the present coordinates inside the sphere.

Also the metric of the space– time outside the sphere is taken to be in the form

$$ds^2 = \frac{1}{\left(A_1 + \frac{A_2}{R}\right)^2} c^2 dT^2 - \left(A_1 + \frac{A_2}{R}\right)^2 \left(dR^2 + R^2 d\overline{\Omega}^2\right) \tag{2}$$

so that the metric is isotropic in space coordinates.

Since the metric has to be Lorentzian at infinity, the constant A_1 should be equal to one ,i.e .$A_1 = 1$.

In general the coordinates in and outside of the sphere do not need to be the same and we take this to be our dissociating point from the work that has been carried out previously until now. Intuitively Seneviratne and de Silva[3] have also used different coordinates in their work.

Therefore in this approach $r = a$ in the matter-filled region corresponds to $R = A$ in the region without matter, outside the sphere.

A new set of boundary conditions has been introduced as

$$\left(\sqrt{g_{ii}}\delta x^i\right)_{r=a} = \left(\sqrt{G_{ii}}\delta X^i\right)_{R=A} \tag{3}$$

$$\left(\frac{1}{\sqrt{g_{ii}}}\left(\frac{d}{dx^i}\left(\sqrt{g_{ii}}\right)\right)\delta x^i\right)_{r=a} = \left(\frac{1}{\sqrt{G_{ii}}}\left(\frac{d}{dX^i}\left(\sqrt{G_{ii}}\right)\right)\delta X^i\right)_{R=A} \tag{4}$$

instead of standard boundary conditions which says that metric coefficients and their partial derivatives are continuous across the boundary.

Here g_{ii} , x^i $i = 0,1,2,3$ and G_{ii} , X^i $i = 0,1,2,3$ are metric coefficients and coordinates for the inside and the outside of the sphere, respectively.

The boundary condition (3) is introduced guiding by the notion of what may be called proper distances and proper times of two observers on either side of the sphere.

The boundary condition (4) is introduced replacing ordinary partial derivatives $\frac{\partial}{\partial r}, \frac{\partial}{\partial \Theta}, \frac{\partial}{\partial \Phi}$ by generalized partial derivatives in curvilinear coordinates in the form

$$\left(\frac{1}{\sqrt{g_{ii}}}\left(\frac{d}{dx^i}\left(\sqrt{g_{ii}}\right)\right)\delta x^i\right)_{r=a} = \left(\frac{1}{\sqrt{G_{ii}}}\left(\frac{d}{dX^i}\left(\sqrt{G_{ii}}\right)\right)\delta X^i\right)_{R=A}$$

These boundary conditions have not been used previously by any other authors, but have been used intuitively by Seneviratne and de Silva [3].

First let us write the metrics in (1) at the boundary $r = a$ and $R = A$ as

$$\delta s^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2} c^2 \delta t^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2 \left(\delta r^2 + r^2(\delta\Theta^2 + \sin^2\Theta\ \delta\Phi^2)\right)$$

$$\delta s^2 = \frac{1}{\left(A_1 + \frac{A_2}{R}\right)^2} c^2 \delta T^2 - \left(A_1 + \frac{A_2}{R}\right)^2 \left(\delta R^2 + R^2\left(\delta\overline{\Theta}^2 + \sin^2\overline{\Theta}\ \delta\overline{\Phi}^2\right)\right) \tag{5}$$

From isotropy Θ has to be equal to $\overline{\Theta}$ and Φ has to be equal to $\overline{\Phi}$ which implies that $\overline{\Omega} = \Omega$.

Hence the metrics (1) take the form

$$ds^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2} c^2 dt^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2 (dr^2 + r^2 d\Omega^2) \qquad 0 \le r \le a$$

$$ds^2 = \frac{1}{\left(1 + \frac{A_2}{R}\right)^2} c^2 dT^2 - \left(1 + \frac{A_2}{R}\right)^2 (dR^2 + R^2 d\Omega^2) \qquad A < R \tag{6}$$

Hence (5) now takes the form

$$\delta s^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2} c^2 \delta t^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2 \left(\delta r^2 + r^2(\delta\Theta^2 + \sin^2\Theta\ \delta\Phi^2)\right)$$

$$\delta s^2 = \frac{1}{\left(1 + \frac{A_2}{R}\right)^2} c^2 \delta T^2 - \left(1 + \frac{A_2}{R}\right)^2 \left(\delta R^2 + R^2(\delta\Theta^2 + \sin^2\Theta\ \delta\Phi^2)\right) \tag{7}$$

at the boundary of the sphere.

Application of the first boundary condition (3) for t and T on the boundary ($r = a$ and $R = A$) gives

$$\frac{1}{\theta\left(\frac{a}{l}\right)} c\ \delta t = \frac{1}{\left(1 + \frac{A_2}{A}\right)} c\ \delta T$$

$$\frac{\delta t}{\delta T} = \frac{\theta\left(\frac{a}{l}\right)}{\left(1 + \frac{A_2}{A}\right)} \tag{8}$$

Application of the second boundary condition (4) for t and T on the boundary ($r = a$ and $R = A$) gives

$$\frac{1}{\theta\left(\frac{a}{l}\right)} \left(-\frac{1}{\left(\theta\left(\frac{a}{l}\right)\right)^2} \frac{1}{l} \theta'\left(\frac{a}{l}\right)\right) c\ \delta t = \frac{1}{\left(1 + \frac{A_2}{A}\right)} \left(-\frac{1}{\left(1 + \frac{A_2}{A}\right)^2} \left(-\frac{A_2}{A^2}\right)\right) c\ \delta T$$

$$\frac{\delta t}{\delta T} = \frac{-A_2 l \left(\theta\left(\frac{a}{l}\right)\right)^3}{A^2 \theta'\left(\frac{a}{l}\right)\left(\left(1 + \frac{A_2}{A}\right)\right)^3} \tag{9}$$

where " $'$ " denotes the differentiation with respect to r .

Application of the first boundary condition (3) for r and R on the boundary ($r = a$ and $R = A$) gives

$$\theta\left(\frac{a}{l}\right)\delta r = \left(1 + \frac{A_2}{A}\right)\delta R$$

$$\frac{\delta r}{\delta R} = \frac{\left(1+\frac{A_2}{A}\right)}{\theta\left(\frac{a}{l}\right)} \tag{10}$$

Application of the second boundary condition (4) for r and R on the boundary ($r = a$ and $R = A$) gives

$$\frac{1}{\theta\left(\frac{a}{l}\right)}\left(\frac{1}{l}\theta'\left(\frac{a}{l}\right)\right)\delta r = \frac{1}{\left(1+\frac{A_2}{A}\right)}\left(-\frac{A_2}{A^2}\right)\delta R$$

$$\frac{\delta r}{\delta R} = \frac{\theta\left(\frac{a}{l}\right)}{\left(1+\frac{A_2}{A}\right)}\left(\frac{-A_2 l}{A^2\theta'\left(\frac{a}{l}\right)}\right) \tag{11}$$

Similarly application of the same boundary conditions (3) and (4) for the angular coordinate Ω in the metric (7) on the boundary ($r = a$ and $R = A$) gives

$$\theta\left(\frac{a}{l}\right)a = \left(1 + \frac{A_2}{A}\right)A \tag{12}$$

$$\frac{1}{\theta\left(\frac{a}{l}\right)}\left(\frac{a}{l}\theta'\left(\frac{a}{l}\right) + \theta\left(\frac{a}{l}\right)\right) = \frac{1}{\left(1+\frac{A_2}{A}\right)}\left(\left(1 + \frac{A_2}{A}\right) + \left(-\frac{A_2}{A^2}A\right)\right) \tag{13}$$

Using the equations (8) and (9) , the constant A_2 can be obtained as

$$\frac{\theta\left(\frac{a}{l}\right)}{\left(1+\frac{A_2}{A}\right)} = \frac{-A_2 l\left(\theta\left(\frac{a}{l}\right)\right)^3}{A^2\theta'\left(\frac{a}{l}\right)\left(\left(1+\frac{A_2}{A}\right)\right)^3}$$

i.e $$A_2 = \frac{\frac{A^2}{l}\left(1+\frac{A_2}{A}\right)^2\theta'\left(\frac{a}{l}\right)}{\left(\theta\left(\frac{a}{l}\right)\right)^2} \tag{14}$$

The equation (12) can be rewritten in the form

$$\frac{\left(1+\frac{A_2}{A}\right)}{\theta\left(\frac{a}{l}\right)} = \frac{a}{A} \tag{15}$$

Then using equation (15), equation (14) can be simplified in to the form

$$A_2 = -\frac{A^2}{l}\left(\frac{a^2}{A^2}\right)\theta'\left(\frac{a}{l}\right) = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right) \tag{16}$$

Substitution of the value of A_2 in equation (12) gives the value of A as

$$A = a\,\theta\left(\frac{a}{l}\right) + \frac{a^2}{l}\theta'\left(\frac{a}{l}\right) \tag{17}$$

Then the metrics (6) become

$$ds^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2}c^2dt^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2(dr^2 + r^2d\Omega^2) \qquad 0 \le r \le a$$

$$ds^2 = \frac{1}{\left(1-\frac{1}{R}\left(\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)\right)\right)^2}c^2dT^2 - \left(1 - \frac{1}{R}\left(\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)\right)\right)^2(dR^2 + R^2d\Omega^2) \qquad A < R \tag{18}$$

where we have replaced A by $(a\,\theta\left(\frac{a}{l}\right) + \frac{a^2}{l}\theta'\left(\frac{a}{l}\right))$.

The values of A which is the radial coordinate just outside the sphere can be found using the values of $\frac{a}{l}$.

The relationship between $\frac{a}{l}$ and A is tabulated in the Table 1 given below.

Table 1: the relationship between $\frac{a}{l}$ and A

$\frac{a}{l}$	$A = (a\,\theta\left(\frac{a}{l}\right) + \frac{a^2}{l}\theta'\left(\frac{a}{l}\right))$
1	0.602929 l
2	0.119738 l
3	-0.57877 l
4	-1.08588 l
5	-1.44905 l
6	-1.75514 l
7	-2.04842 l

Substitution of the values for A_2 and A in (8) gives

$$\frac{\delta t}{\delta T} = \theta\left(\frac{a}{l}\right) + \frac{a}{l}\theta'\left(\frac{a}{l}\right) \tag{19}$$

The values of $\frac{\delta t}{\delta T}$ for few different values of $\frac{a}{l}$ has been tabulated in Table 2 as given below.

Table 2: Few values of $\frac{\delta t}{\delta T}$

$\frac{a}{l}$	$\frac{\delta t}{\delta T} = \theta\left(\frac{a}{l}\right) + \frac{a}{l}\theta'\left(\frac{a}{l}\right)$
1	0.602929
2	0.059869
3	-0.19292
4	-0.27139
5	-0.28981
6	-0.29252
7	-0.29252

Since δt increases as , $\frac{\delta t}{\delta T}$ should be positive ,i.e. $\frac{\delta t}{\delta T} > 0$.

Hence according to Table 2, there is a maximum value that $\frac{a}{l}$ can take, which is nearly 2.16 .

Figure 1 which have the graphs of $\theta\left(\frac{R}{l}\right)$ and $\theta\left(\frac{R}{l}\right) + \frac{R}{l}\theta'\left(\frac{R}{l}\right)$ versus $\frac{R}{l}$ in Sashinka wimaladharma and Nalin de Silva[1] also confirms this result and a more accurate value for the maximum of $\frac{a}{l}$ is 2.157 as obtained using the graph in Figure 1 in Sashinka wimaladharma and Nalin de Silva [1].

Considering the values of A in Table 1, it can be found that they are positive only when $\frac{a}{l} \leq 2.16$ as A differs from $\frac{\delta t}{\delta T}$ by a factor of a which is positive being the radial coordinate as measured inside the sphere at its surface.

Therefore spheres whose radial coordinate is greater than $2.16\, l$ with suitable units, as measured just inside the spheres can not be exist.

Now the metric (18) outside the sphere can be written as

$$ds^2 = \frac{1}{\left(1-\frac{1}{R}\left(\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)\right)\right)^2} c^2 dT^2 - \left(1 - \frac{1}{R}\left(\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)\right)\right)^2 \left(dR^2 + R^2 d\overline{\Omega}^2\right)$$

But in Wickramasuriya and Bonnar [5] , it has been shown that $m = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)$ where m is the mass of the sphere.

Hence the metric (18) outside the sphere takes the form

$$ds^2 = \frac{1}{\left(1+\frac{m}{R}\right)^2} c^2 dT^2 - \left(1+\frac{m}{R}\right)^2 (dR^2 + R^2 d\Omega^2)$$

Thus introducing different coordinates in the regions inside and outside the sphere gives us a metric which is Lorentzian at infinity, unlike in Sashinka wimaladharma and Nalin de Silva [1] where was a metric that was not Lorenzian at infinity.

Now considering the metric (1) without initially assuming that it becomes Lorentzian at infinity, and applying the same boundary conditions as above, it can be shown that the value of the constant A_1 and A_2 are such that

$$A_1 = \frac{a}{A}\left(\theta\left(\frac{a}{l}\right) + \frac{a}{l}\theta'\left(\frac{a}{l}\right)\right) \tag{20}$$

$$A_2 = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right). \tag{21}$$

Thus it is found that the value for A_2 is the same for the metrics obtained in Sashinka wimaladharma and Nalin de Silva [1] , (16) and (21) irrespective of whether the metric is Lorentzian at infinity.

This could have been expected as the expression for A_2 gives the value of *m,* the mass of the distribution as calculated inside the sphere by integration as in Wickramasuriya and Bonnar [4] .

If the metric for the outside of the sphere is Lorentzian , then then the constant A_1 should be equal to unity, i.e $A_1 = \frac{a}{A}\left(\theta\left(\frac{a}{l}\right) + \frac{a}{l}\theta'\left(\frac{a}{l}\right)\right) = 1$ which implies that $A = a\left(\theta\left(\frac{a}{l}\right) + \frac{a}{l}\theta'\left(\frac{a}{l}\right)\right)$ which is the value of the radial coordinate at the surface of the sphere as measured just outside the sphere.

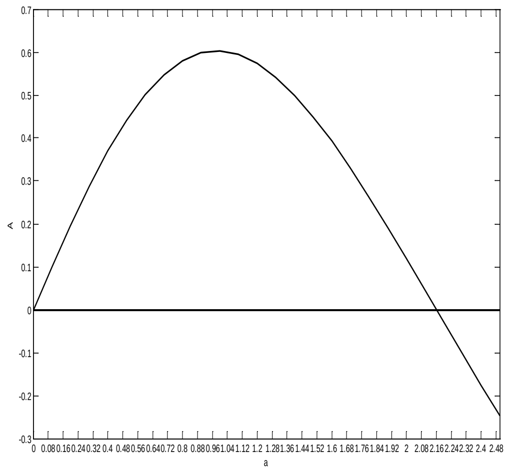
As before A becomes zero when $\frac{a}{l}$ is nearly equal to 2.16. Thus $\frac{a}{l}$ has a maximum value nearly equal to 2.16.

The graph of AA_1 against $\frac{a}{l}$ is plotted in Figure 1. AA_1

It can be found from the Figure 1 that AA_1 has a maximum value which is equal to 0.6 l.

Therefore when the metric is Lorentzian $(A_1 = 1)$ the maximum value of A equal to 0.6 l.

This corresponds to a value a which is equal to l. Thus the outer radial coordinate A increases with inner radial coordinate a until 0.6 l . After that A decreases with a and becomes zero when $a = 2.16\ l$.



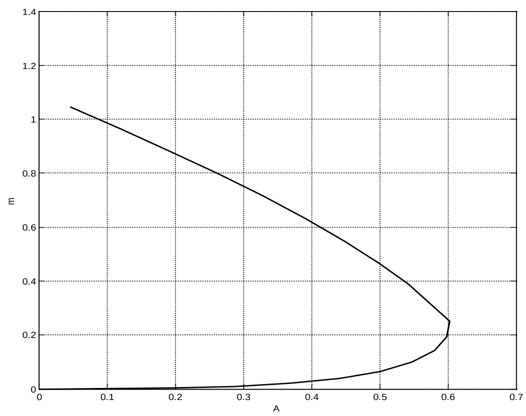
$\frac{a}{l}$

Figure 1: the graph of AA_1 against $\frac{a}{l}$

It can be postulated that A increases and attains a maximum at $A = 0.6\,l$ when $a = l$, and that there is no decrease of A as a increases though mathematically it is suggested so. If A decreases with a after that it would end up with zero and then with negative values that have no meaning physically.

The mass of the sphere which is equal to $m = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)$ is plotted against A in the Figure 2.

$$m = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)$$



A

Figure 2: the graph of m against A

Figure 2 suggests that m increases with A until reaches the value $A = 0.6\,l$ .After that according to the Figure 2, A decreases though m increases , and it can be postulated that m has a maximum 1.12 l when $A = 0.6\,l$.

The maximum value of A is the same for both cases that obtained from Figure 1 and Figure 2.

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The graph of $m = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)$ versus $\frac{a}{l}$ is as shown in Figure 3.

$m = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right)$

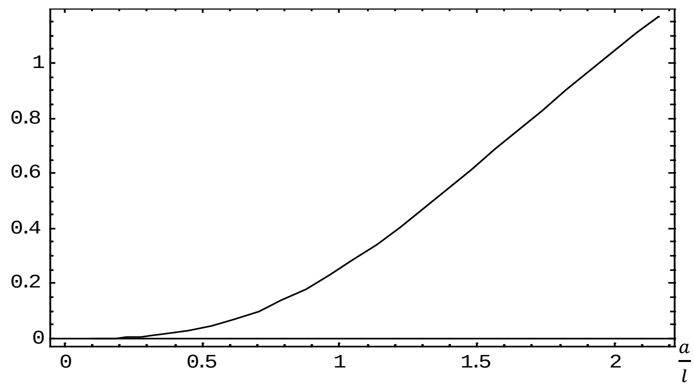

Figure 3: the graph of m against $\frac{a}{l}$

In Figure 3 , m increases with $\frac{a}{l}$ without a maximum. However a has a maximum $2.16\ l$ and the corresponding value of m is $1.12\ l$.

Hence it can be postulated that $1.12\ l$ is the maximum value that m can take.

This is an interesting result as in the case of electrically counterpoised dust distribution , there are maximum values that are written in terms of l which can in turn be expressed in terms of density as $4\pi\rho l^2 = 1$.

This implies that given the constant density of the distribution there are maximum values that

m, a (and hence A) can take.

3. Results

3.1. The Red Shift of a Pulse of Light

An expression for the red shift of a pulse of light emitted at a point inside of the sphere along a radial direction as an observer who is at a large distance away from the sphere has been calculated.

However in this case, unlike in the case in Sashinka Wimaladharma and Nalin de silva [1] where we used standard Lichernowicz boundary conditions[5], the differences in time when a light ray passes through the boundary of the sphere have taken into consideration.

First a pulse of light with front emitted at $r = r_e$ at $t = t_e$ with frequency ν_e inside the sphere, and an observer at $r = a$, just inside the boundary of the sphere receiving it at $t = t'_e$ with frequency ν'_e has been considered.

The radial null geodesics within the sphere are given by the metric (18) as

$$0 = \frac{1}{\left(\theta\left(\frac{r}{l}\right)\right)^2} c^2 dt^2 - \left(\theta\left(\frac{r}{l}\right)\right)^2 dr^2 \,,$$

Taking the plus sign since r increases with time t as the photons are going away from the centre of the sphere gives

$$\frac{dr}{dt} = \frac{c}{\left(\theta\left(\frac{r}{l}\right)\right)^2}. \tag{28}$$

Equation (28) can be integrated considering the front of the pulse as

$$\int_{t_e}^{t'_e} c\, dt = \int_{r_e}^{a} \left(\theta\left(\frac{r}{l}\right)\right)^2 dr \tag{29}$$

Assuming that the rear of the pulse which is emitted at $r = r_e$ at $t = t_e + \Delta t_e$ with frequency ν_e will observe it at $= a$, just inside the boundary of the sphere at $t = t'_e + \Delta t'_e$ with frequency ν'_e,

Equation (28) can also be integrated considering the rear of the pulse as

$$\int_{t_e+\Delta t_e}^{t'_e+\Delta t'_e} c\, dt = \int_{r_e}^{a} \left(\theta\left(\frac{r}{l}\right)\right)^2 dr \tag{30}$$

Since the right hand sides of equations (29) and (30) are the same , it can be obtained that

$$\int_{t_e}^{t'_e} c\, dt = \int_{t_e+\Delta t_e}^{t'_e+\Delta t'_e} c\, dt \,. \tag{31}$$

Rearrangement and simplification of the equation (31) with the assumption of Δt_e and $\Delta t'_e$ are very small gives

$$\Delta t_e = \Delta t'_e \tag{32}$$

Now the proper time intervals from the equation (18) of the two observers corresponding to Δt_e and $\Delta t'_e$ are given by

$$\Delta\tau_e = \left(\frac{1}{\theta\left(\frac{r_e}{l}\right)}\right) \Delta t_e \quad \text{and} \quad \Delta\tau'_e = \left(\frac{1}{\theta\left(\frac{a}{l}\right)}\right) \Delta t'_e \tag{33}$$

where $\Delta\tau_e$ and $\Delta\tau'_e$ are the proper time intervals of the two observers at $r = r_e$ and $r = a$, just inside the boundary of the sphere , respectively, emitting and receiving the pulse.

The number of cycles of the pulse remain the same at emission and observation yields

$$\nu_e \Delta\tau_e = \nu'_e \Delta\tau'_e$$

$$\frac{\nu_e'}{\nu_e} = \frac{\Delta\tau_e}{\Delta\tau_e'}$$

(34)

Now equation (33) and (34) give $\frac{\nu_e'}{\nu_e} = \frac{\theta\left(\frac{a}{l}\right)}{\theta\left(\frac{r_e}{l}\right)}\frac{\Delta t_e}{\Delta t_e'}$ (35)

Using equation (32), equation (35) can be simplified to

$$\frac{\nu_e'}{\nu_e} = \frac{\theta\left(\frac{a}{l}\right)}{\theta\left(\frac{r_e}{l}\right)} \tag{36}$$

Now consider the moment which the pulse of ray pass through the boundary of the sphere.

It is assumed that an observer at $R = A$ but outside of the sphere observed the pulse at $T = T_0'$ with frequency ν_0'.

Then the relationship of the time intervals $\Delta t_e'$ and $\Delta T_0'$ is given by the equation (19), which is

$$\frac{\Delta t_e'}{\Delta T_0'} = \theta\left(\frac{a}{l}\right) + \frac{a}{l}\theta'\left(\frac{a}{l}\right) \tag{37}$$

Then using the metrics in (18), the proper times of the two relevant observers corresponding to $\Delta t_e'$ and $\Delta T_0'$ are given by

$$\Delta\tau_e' = \left(\frac{1}{\theta\left(\frac{a}{l}\right)}\right)\Delta t_e' \quad \text{and} \quad \Delta\tau_0' = \left(\frac{1}{1-\frac{a^2\theta'\left(\frac{a}{l}\right)}{lA}}\right)\Delta T_0' \tag{38}$$

where $\Delta\tau_e'$ and $\Delta\tau_0'$ are the proper time intervals of the two observers at $r = a$,inside the sphere and $R = A$,outside of the sphere respectively receiving the pulse and A is given by

$$A = a\theta\left(\frac{a}{l}\right) + \frac{a^2}{l}\theta'\left(\frac{a}{l}\right).$$

Using the similar steps in equation (34),(35) and (36), the ratio between the frequencies ν_e' and ν_0' can be obtained in the form

$$\frac{\nu_e'}{\nu_0'} = 1 \tag{39}$$

Now assuming that an observer at a large distance $R = R_0$, outside of the sphere observed the front of the pulse which is passed through the boundary at $T = T_0'$ outside the sphere, the radial null geodesic for the exterior vacuum region can be integrated into the form

$$\int_{T_0'}^{T_0} cdT = \int_A^{R_0} \left(1 - \frac{a^2\theta\prime\left(\frac{a}{l}\right)}{lA}\right)^2 dR \tag{40}$$

If the rear of the pulse is observed at $T = T_0' + \Delta T_0'$ at $R = R_0$, then the radial null geodesic for the exterior vacuum region can be integrated into the form

$$\int_{T_0'+\Delta T_0'}^{T_0+\Delta T_0} cdT = \int_A^{R_0} \left(1 - \frac{a^2\theta\prime\left(\frac{a}{l}\right)}{lA}\right)^2 dR \tag{41}$$

Since the right hand sides of (40) and (41) are equal and hence it can be obtained

$$\int_{T_0'}^{T_0} cdT = \int_{T_0'+\Delta T_0'}^{T_0+\Delta T_0} cdT \ .$$

Rearrangement and simplification with the assumption that ΔT_0 and $\Delta T_0'$ are very small gives

$$\Delta T_0 = \Delta T_0' \tag{42}$$

Now using the metrics in (18), the proper times of the two observers corresponding to $\Delta T_0'$ and ΔT_0 can be written as

$$\Delta \tau_0' = \left(\frac{1}{1-\frac{a^2\theta\prime\left(\frac{a}{l}\right)}{lA}}\right)\Delta T_0' \qquad \text{and} \qquad \Delta \tau_0 = \left(\frac{1}{1-\frac{a^2\theta\prime\left(\frac{a}{l}\right)}{lR_0}}\right)\Delta T_0 \tag{43}$$

where $\Delta \tau_0'$ and $\Delta \tau_0$ are the proper time intervals of the two observers at A,just outside the boundary of the sphere and at $R = R_0$, a large distance outside of the sphere respectively receiving the pulse.

Using the fact that the number of cycles of the pulse remain the same throughout its way and equation (43) , the ratio between the frequencies ν_0 and ν_0' can be obtained as

$$\frac{\nu_0}{\nu_0'} = \frac{\left(1-\frac{a^2\theta\prime\left(\frac{a}{l}\right)}{lR_0}\right)}{\left(1-\frac{a^2\theta\prime\left(\frac{a}{l}\right)}{lA}\right)}\frac{\Delta T_0}{\Delta T_0'} \tag{44}$$

Equation (44) can be simplified using equation (42) into the form

$$\frac{\nu_0}{\nu_0'} = \frac{\left(1-\frac{a^2\theta\prime\left(\frac{a}{l}\right)}{lR_0}\right)}{\left(1-\frac{a^2\theta\prime\left(\frac{a}{l}\right)}{lA}\right)} \tag{45}$$

Using the fact that $\frac{\nu_0}{\nu_e} = \frac{\nu_0}{\nu_0'}\frac{\nu_0'}{\nu_e'}\frac{\nu_e'}{\nu_e}$ and equations (36),(39) and (45) , it can be obtained that

$$\frac{\nu_0}{\nu_e} = \left(\frac{1 - \frac{a^2\theta'\left(\frac{a}{l}\right)}{lR_0}}{1 - \frac{a^2\theta'\left(\frac{a}{l}\right)}{lA}} \right)(1)\left(\frac{\theta\left(\frac{a}{l}\right)}{\theta\left(\frac{r_e}{l}\right)} \right) \tag{46}$$

Since the red shift z corresponds to the change of the wave length can be calculated as

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{\left(1 - \frac{a^2\theta'\left(\frac{a}{l}\right)}{lA}\right)\theta\left(\frac{r_e}{l}\right)}{\left(1 - \frac{a^2\theta'\left(\frac{a}{l}\right)}{lR_0}\right)\theta\left(\frac{a}{l}\right)}$$

When R_0 tends to infinity, the red shift is equal to $1 + z = \frac{\left(1 - \frac{a^2\theta'\left(\frac{a}{l}\right)}{lA}\right)\theta\left(\frac{r_e}{l}\right)}{\theta\left(\frac{a}{l}\right)}$.

When using standard boundary conditions in Sashinka Wimaladharma and Nalin de Silva[1],

the red shift was calculated to be $1 + z = \frac{\theta\left(\frac{r_e}{l}\right)}{\left(\theta\left(\frac{a}{l}\right) - \frac{m}{a}\right)}$.

The red shifts as observed at infinity are all equal provided that $A = -\frac{a^3\theta'\left(\frac{a}{l}\right)}{ml}\left(\theta\left(\frac{a}{l}\right) - \frac{m}{a}\right)$.

Going by the previous results we may impose this condition on A so that the red shift is an invariant with respect to change of coordinates.

4. Conclusion

A static spherically symmetric solution to Einstein-Maxwell field equations for a spherical distribution of electrically counterpoised dust distribution has been found using a new set of boundary conditions with the assumption that the coordinates in inside and outside of the sphere are different each other. Also the new metric is Lorentzian everywhere.

It is also found that the mass m, radial coordinate just inside the boundary a and radial coordinate just outside of the boundary A have maximum values. Therefore Einstein's equations have been solved for finite distribution of matter.

Imposing the condition $A = -\frac{a^3\theta'\left(\frac{a}{l}\right)}{ml}\left(\theta\left(\frac{a}{l}\right) - \frac{m}{a}\right)$ on A, it can be shown that the red shift is an invariant with respect to change of coordinates.

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