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On Study K^h Generalized Birecurrent Affinely Connected Space

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Abstract

In the present paper we introduce a K^h – *generalized birecurrent space* which characterized by the condition $K^i_{jkh|\ell|m} = \lambda_\ell K^i_{jkh|m} + b_{\ell m} K^i_{jkh}$, $K^i_{jkh} \neq 0$, where λ_ℓ and $b_{\ell m}$ are non-zero covariant vector fields and covariant tensor field of second order, respectively. This space satisfies the condition of affinely connected space called K^h – *generalized birecurrent affinely connected space*.

Keywords: Finsler space; $K^h -$ Generalized birecurrent space; Ricci tensor.

1. Introduction

H. D. Pande and B. Single [4] discussed the recurrence property in an affinely connected space. P. K. Dwivedi [7] worked out the role of P^* – reducible space in affinely connected space. A. A. M. Saleem [2] obtained some results when the C^h – *generalized birecurrent* and C^h – *special generalized birecurrent* are affinely connected spaces. A. A. A. Muhib [1] obtained some results when R^h – *generalized trirecurrent and* R^h – *special generalized trirecurrent* are affinely connected spaces. M. A. A. Ali [5] obtained certain identities in a K^h – *birecurrent* affinely connected spaces. N. S. H. Hussien [6] obtained certain identities in a K^h – *recurrent* affinely connected spaces.

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Let F_n be an n-dimensional Finsler space equipped with the metric function a $F(x,y)$ satisfying the request conditions [3].

The vectors y_i , y^i and the metric tensor g_{ij} satisfies the following relations

(1.1) a) $y^i_{\ |k} = 0$ and b) $g_{ij|k} = 0$,

The $h-$ covariant differentiation with respect to x^k, commute with the partial differentiation with respect to y^j according to

(1.2) a) $\dot{\partial}_j(X^i_{\ |k}) - (\dot{\partial}_j X^i)_{|k} = X^r(\dot{\partial}_j \Gamma^{*i}_{rk}) - (\dot{\partial}_r X^i) P^r_{jk}$,

where

(1.2) b) $P^r_{jk} := (\dot{\partial}_j \Gamma^{*r}_{hk}) y^h = \Gamma^{*r}_{jhk} \ y^h$

The tensor K^i_{rkh} is called *Cartan's fourth curvature tensor* which is skew-symmetric in its last two lower indices k and h , i. e.

(1.3) $K^i_{jkh} = - K^i_{jhk}$.

The associate tensor K_{ijkh} of the curvature tensor K^i_{jkh} is given by

(1.4) $K_{ijkh} := g_{rj} K^r_{ikh}$.

The Ricci tensor K_{jk} of the curvature tensor K^i_{jkh} is given by

(1.5) $K^i_{jki} = K_{jk}$.

The curvature tensor K^i_{jkh} satisfies the following relations too

(1.6) $K^i_{jkh} \ y^j = H^i_{kh}$

and

(1.7) $H^i_{jkh} - K^i_{jkh} = P^i_{jk|h} + P^r_{jk} P^i_{rh} - h/k$.

Berwald curvature tensor H^i_{jkh} satisfies the relation

(1.8) $H^i_{jkh} \, y^j = H^i_{kh}$

and

(1.9) $H^i_{jkh} = \dot{\partial}_j H^i_{kh}$,

where $H^{\,i}_{kh}$ called $h(v)-$ torsion tensor.

Also, satisfies bianchi identity

(1.10) a) $H^i_{jkh} + H^i_{hjk} + H^i_{khj} = 0$

and it is skew- symmetric in its last two lower indices, i.e.

(1.10) $b)$ $H^i_{jkh} = -H^i_{jhk}$.

The deviation tensor $H^{\,i}_{k}$ is positively homogeneous of degree two in $y^{\,i}$ and satisfies

(1.11) $H^i_{hk}\, y^{\,h} = H^i_k$,

(1.12) $H_{jk} = H^i_{jki}$,

(1.13) $H_k = H^i_{ki}$,

and

(1.14) $H = \frac{1}{n-1}\ H^i_i$,

where H_{jk} and H are called *h-Ricci tensor* and *curvature scalar*, respectively. Since contraction of the indices does not affect the homogeneity in y^i, hence the tensors H_{rk} , H_r and the scalar H are also homogeneous of degree zero, one and two in y^i, respectively.

The associate tensor H_{ijkh} of Berwald curvature tensor H^i_{jkh} is given by

(1.15) $H_{ijkh} = g_{rj}\, H^r_{ikh}$.

The contraction of the indices i and j in (1.10a) and by using (1.12) and the skew-symmetric property of the curvature tensor $H^{\,i}_{jkh}$ in the last two lower indices, shows that the $h-$ Ricci tensor satisfies

(1.16) $H^r_{rkh} = H_{hk} - H_{kh}$.

The tensor $H_{jh.k}$ defined by

(1.17) $H_{jh.k} := g_{ih} H^i_{jk}$.

Cartan's fourth curvature tensor $K^{\,i}_{jkh}$ satisfies the following identity known as *Bianchi identity*

(1.18) $K^i_{jkh|\ell} + K^i_{j\ell k|h} + K^i_{jh\ell|k} + y^r\{(\dot{\partial}_s \Gamma^{*i}_{jk}) K^s_{rh\ell} + (\dot{\partial}_s \Gamma^{*i}_{j\ell}) K^s_{rkh} +$

$(\dot{\partial}_s \Gamma^{*i}_{jh}) K^s_{r\ell k}\} = 0.$

A Finsler space whose Berwald's connection parameter G^i_{jk} is independent of y^i is called *an affinely connected space (Berwald space).* Thus, an affinely connected space has some properties as follows:

(1.19) $G^i_{jkh} = 0$

and

(1.20) $C_{ijk|h} = 0$.

The connection parameters Γ^{*i}_{jk} of Cartan and G^i_{jk} of Berwald coincide in an affinely connected space and they are independent of the direction arguments [3], i.e.

(1.21) $G^i_{jkh} = \dot{\partial}_j G^i_{kh} = 0$

and

(1.22) $\dot{\partial}_j \Gamma^{*i}_{kh} = 0.$

N. S. H. Hussein [3] introduce the K^h–*recurrent space* which characterized by the condition

(1.23) $K^i_{jkh|\ell} = \lambda_\ell K^i_{jkh}$, $K^i_{jkh} \neq 0,$

where the covariant vector field λ_ℓ being the recurrence vector field.

2. An K^h − Generalized Birecurrent Space

Let us consider a Finsler space F_n whose Cartan's fourth curvature tensor K^i_{jkh} satisfies the condition

(2.1) $K^i_{jkh|\ell|m} = \lambda_\ell K^i_{jkh|m} + b_{\ell m} K^i_{jkh}$, $K^i_{jkh} \neq 0,$

where λ_ℓ and $b_{\ell m}$ are non-zero covariant vector field and covariant tensor field of second order, respectively. The space satisfying the condition (2.1) will be called K^h–*generalized birecurrent space.* We shall denote it briefly by K^h– GBR– F_n .

Transvecting (2.1) by y^j, using (1.1a) and (1.6), we get

(2.2) $H^i_{kh|\ell|m} = \lambda_\ell H^i_{kh|m} + b_{\ell m} H^i_{kh}$.

Theorem 2.1. In $K^h - GBR - F_n$, *the* $h(v)-$ *torsion tensor* H^i_{kh} *is generalized birecurrent.*

Differentiating (2.2) partially with respect to y^j and using (1.9), we get

(2.3) $$\dot\partial_j H^i_{kh|\ell|m} = (\dot\partial_j \lambda_\ell) H^i_{kh|m} + \lambda_\ell \dot\partial_j (H^i_{kh|m}) + (\dot\partial_j b_{\ell m}) H^i_{kh} + b_{\ell m} H^i_{jkh}.$$

Using the commutation formula exhibited by (1.2a) for $(H^i_{kh|\ell})$ and (H^i_{kh}) in (2.3) and using (1.9), we get

(2.4) $$\{ \dot\partial_j (H^i_{kh|\ell}) \}_{|m} + H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|\ell}(\dot\partial_j \Gamma^{*r}_{hm})$$

$$- H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{\ell m}) - \dot\partial_r (H^i_{kh|\ell}) P^r_{jm} = (\dot\partial_j \lambda_\ell) H^i_{kh|m} + \lambda_\ell H^i_{jkh|m} +$$

$$\lambda_\ell [H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{rkh} P^r_{jm}] +$$

$$(\dot\partial_j b_{\ell m}) H^i_{kh} + b_{\ell m} H^i_{jkh}.$$

Again applying the commutation formula exhibited by (1.2a) for (H^i_{kh}) in (2.4) and using (1.9), we get

$$\{ H^i_{jkh|\ell} + H^r_{kh}(\dot\partial_j \Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^i_{kr}(\dot\partial_j \Gamma^{*r}_{h\ell}) - H^i_{rkh} P^r_{j\ell} \}_{|m}$$

$$+ H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|\ell}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{\ell m}) -$$

$$\{ H^i_{rkh|\ell} + H^s_{kh}(\dot\partial_r \Gamma^{*i}_{s\ell}) - H^i_{sh}(\dot\partial_r \Gamma^{*s}_{k\ell}) - H^i_{ks}(\dot\partial_r \Gamma^{*s}_{h\ell}) - H^i_{skh} P^s_{r\ell} \} P^r_{jm}$$

$$= (\dot\partial_j \lambda_\ell) H^i_{kh|m} + \lambda_\ell H^i_{jkh|m} + \lambda_\ell [H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{km})$$

$$- H^i_{kr}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{rkh} P^r_{jm}] + (\dot\partial_j b_{\ell m}) H^i_{kh} + b_{\ell m} H^i_{jkh}$$

which can be written as

(2.5) $$H^i_{jkh|\ell|m} + \left\{ H^r_{kh}(\dot\partial_j \Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^i_{kr}(\dot\partial_j \Gamma^{*r}_{h\ell}) - H^i_{rkh} P^r_{j\ell} \right\}_{|m} +$$

$$H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|\ell}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{\ell m}) - H^i_{rkh|\ell} P^r_{jm}$$

$$- H^s_{kh}(\dot\partial_r \Gamma^{*i}_{s\ell}) P^r_{jm} + H^i_{sh}(\dot\partial_r \Gamma^{*s}_{k\ell}) P^r_{jm} + H^i_{ks}(\dot\partial_r \Gamma^{*s}_{h\ell}) P^r_{jm} + H^i_{skh} P^s_{r\ell} P^r_{jm} =$$

$$\lambda_\ell H^i_{jkh|m} + b_{\ell m} H^i_{jkh} + (\dot\partial_j \lambda_\ell) H^i_{kh|m} + \lambda_\ell H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - \lambda_\ell H^i_{rh}(\dot\partial_j \Gamma^{*r}_{km})$$

$$- \lambda_\ell H^i_{kr}(\dot\partial_j \Gamma^{*r}_{hm}) - \lambda_\ell H^i_{rkh} P^r_{jm} + (\dot\partial_j b_{\ell m}) H^i_{kh} .$$

This shows that

$$H^i_{jkh|\ell|m} = \lambda_\ell H^i_{jkh|m} + b_{\ell m} H^i_{jkh}$$

if and only if

(2.6) $\left\{ H^r_{kh}(\dot\partial_j \Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^i_{kr}(\dot\partial_j \Gamma^{*r}_{h\ell}) - H^i_{rkh} P^r_{j\ell} \right\}_{|m} + H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm})$

$- H^i_{rh|\ell}(\dot\partial_j \Gamma^{*i}_{km}) - H^i_{kr|\ell}(\dot\partial_j \Gamma^{*i}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{\ell m}) - H^i_{rkh|\ell} P^r_{jm} -$

$H^s_{kh}(\dot\partial_r \Gamma^{*i}_{s\ell}) P^r_{jm} + H^i_{sh}(\dot\partial_r \Gamma^{*s}_{k\ell}) P^r_{jm} + H^i_{ks}(\dot\partial_r \Gamma^{*s}_{h\ell}) P^r_{jm} + H^i_{skh} P^s_{r\ell} P^r_{jm}$

$= (\dot\partial_j \lambda_\ell) H^i_{kh|m} + \lambda_\ell H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - \lambda_\ell H^i_{rh}(\dot\partial_j \Gamma^{*i}_{km}) - \lambda_\ell H^i_{kr}(\dot\partial_j \Gamma^{*i}_{hm})$

$-\lambda_\ell H^i_{rkh} P^i_{jm} + (\dot\partial_j b_{\ell m}) H^i_{kh}.$

Thus, we conclude

Theorem 2.2. *In $K^h - GBR - F_n$, Berwald curvature tensor H^i_{jkh} is generalized birecurrent if and only if (2.6) holds good.*

Transvecting (2.5) by g_{ip}, using (1.1b) and (1.15), we get

(2.7) $H_{jpkh|\ell|m} + g_{ip}\left[\left\{ H^r_{kh}(\dot\partial_j \Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^i_{kr}(\dot\partial_j \Gamma^{*r}_{h\ell}) - H^i_{rkh} P^r_{j\ell} \right\}_{|m}\right.$

$+ H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|\ell}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{\ell m}) -$

$H^i_{rkh|\ell} P^r_{jm} - H^s_{kh}(\dot\partial_r \Gamma^{*i}_{s\ell}) P^r_{jm} + H^i_{sh}(\dot\partial_r \Gamma^{*s}_{k\ell}) P^r_{jm} + H^i_{ks}(\dot\partial_r \Gamma^{*s}_{h\ell}) P^r_{jm} +$

$\left. H^i_{skh} P^s_{r\ell} P^r_{jm}\right] = (\lambda_\ell H_{jpkh|m} + b_{\ell m} H_{jpkh}) + g_{ip}\{(\dot\partial_j \lambda_\ell) H^i_{kh|m} +$

$\lambda_\ell H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - \lambda_\ell H^i_{rh}(\dot\partial_j \Gamma^{*r}_{km}) - \lambda_l H^i_{kr}(\dot\partial_j \Gamma^{*r}_{hm}) - \lambda_\ell H^i_{rkh} P^r_{jm} + (\dot\partial_j b_{\ell m}) H^i_{kh}\}.$

This shows that

$$H_{jpkh|\ell|m} = \lambda_\ell H_{jpkh|m} + b_{\ell m} H_{jpkh}$$

if and only if

(2.8) $g_{ip}\left[\left\{ H^r_{kh}(\dot\partial_j \Gamma^{*i}_{r\ell}) - H^i_{rh}(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^i_{kr}(\dot\partial_j \Gamma^{*r}_{h\ell}) - H^i_{rkh} P^r_{j\ell} \right\}_{|m}\right.$

$+ H^r_{kh|\ell}(\dot\partial_j \Gamma^{*i}_{rm}) - H^i_{rh|\ell}(\dot\partial_j \Gamma^{*r}_{km}) - H^i_{kr|\ell}(\dot\partial_j \Gamma^{*r}_{hm}) - H^i_{kh|r}(\dot\partial_j \Gamma^{*r}_{\ell m}) -$

$$H^i_{rkh|\ell}P^r_{jm} - H^s_{kh}(\dot\partial_r \Gamma^{*i}_{s\ell})\, P^r_{jm} + H^i_{sh}(\dot\partial_r \Gamma^{*s}_{k\ell})P^r_{jm} + H^i_{ks}(\dot\partial_r \Gamma^{*s}_{h\ell})P^r_{jm} +$$

$$H^i_{skh}P^s_{r\ell}\;\; P^r_{jm}] = g_{ip}\{(\dot\partial_j \lambda_l\,)H^i_{kh|m} + \lambda_l\, H^r_{kh}(\dot\partial_j \Gamma^{*i}_{rm}) - \lambda_l\, H^i_{rh}(\dot\partial_j \Gamma^{*r}_{km})$$

$$-\lambda_l\, H^i_{kr}(\dot\partial_j \Gamma^{*r}_{hm}) - \lambda_l\, H^i_{rkh}P^r_{jm} + (\dot\partial_j b_{lm})H^i_{kh}\}.$$

Thus, we conclude

Theorem 2.3. *In* $K^h\!-GBR\!-F_n$, *the associate tensor* H_{jpkh} *of Berwald curvature tensor* H^i_{jkh} *is generalized birecurrent if and only if (2.8) holds good.*

Contracting the indices i and h in (2.5), using (1.12) and (1.13), we get

(2.9) $$H_{jk|\ell|m} + \{H^r_{kp}(\dot\partial_j \Gamma^{*p}_{r\ell}) - H_r(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^p_{kr}(\dot\partial_j \Gamma^{*r}_{p\ell}) - H_{rk}(\dot\partial_j \Gamma^{*r}_{j\ell})\}_{|m}$$

$$+H^r_{kp|\ell}(\dot\partial_j \Gamma^{*p}_{rm}) - H_{r|\ell}\,(\dot\partial_j \Gamma^{*r}_{km}) - H^p_{kr|\ell}(\dot\partial_j \Gamma^{*r}_{pm}) - H_{k|r}(\dot\partial_j \Gamma^{*r}_{\ell m}) -$$

$$H_{rk|\ell}\, P^r_{jm} - H^s_{kp}(\dot\partial_r \Gamma^{*p}_{s\ell})\, P^r_{jm} + H_s(\dot\partial_r \Gamma^{*s}_{k\ell})P^r_{jm} + H^p_{ks}(\dot\partial_r \Gamma^{*s}_{p\ell})P^r_{jm} +$$

$$H_{sk}\, P^s_{r\ell}\;\; P^r_{jm} = (\lambda_\ell\, H_{jk|m} + b_{\ell m}\, H_{jk}\,) + (\dot\partial_j \lambda_\ell)H_{k|m} + \lambda_\ell H^r_{kp}(\dot\partial_j \Gamma^{*p}_{rm})$$

$$-\lambda_\ell H_r(\dot\partial_j \Gamma^{*r}_{km}) - \lambda_\ell H^p_{kr}(\dot\partial_j \Gamma^{*r}_{pm}) - \lambda_\ell H_{rk}\, P^r_{jm} + (\dot\partial_j b_{\ell m}\,)H_k.$$

This shows that

$$H_{jk|\ell|m} = \lambda_\ell H_{jk|m} + b_{\ell m}\, H_{jk}$$

if and only if

(2.10) $$\{H^r_{kp}(\dot\partial_j \Gamma^{*p}_{r\ell}) - H_r(\dot\partial_j \Gamma^{*r}_{k\ell}) - H^p_{kr}(\dot\partial_j \Gamma^{*r}_{p\ell}) - H_{rk}(\dot\partial_j \Gamma^{*r}_{j\ell})\}_{|m}$$

$$+H^r_{kp|\ell}(\dot\partial_j \Gamma^{*p}_{rm}) - H_{r|\ell}\,(\dot\partial_j \Gamma^{*r}_{km}) - H^p_{kr|\ell}(\dot\partial_j \Gamma^{*r}_{pm}) - H_{k|r}(\dot\partial_j \Gamma^{*r}_{\ell m}) -$$

$$H_{rk|\ell}\, P^r_{jm} - H^s_{kp}(\dot\partial_r \Gamma^{*p}_{s\ell})\, P^r_{jm} + H_s(\dot\partial_r \Gamma^{*s}_{k\ell})P^r_{jm} + H^p_{ks}(\dot\partial_r \Gamma^{*s}_{p\ell})P^r_{jm} +$$

$$H_{sk}\, P^s_{r\ell}\;\; P^r_{jm} = (\dot\partial_j \lambda_\ell)H_{k|m} + \lambda_\ell H^r_{kp}(\dot\partial_j \Gamma^{*p}_{rm}) - \lambda_\ell H_r(\dot\partial_j \Gamma^{*r}_{km})$$

$$\lambda_\ell H^p_{kr}(\dot\partial_j \Gamma^{*r}_{pm}) - \lambda_\ell H_{rk}\, P^r_{jm} + (\dot\partial_j b_{\ell m}\,)H_k\,.$$

Theorem 2.4. *In* $K^h\!-GBR\!-F_n$, *K–Ricci tensor* H_{jk} *in sense of Cartan is generalized birecurrent if and only if (2.10) holds good.*

Contracting the indices i and j in (2.5) and using (1.16), we get

(2.11) $$(H_{hk} - H_{kh})_{|\ell|m} + \{H^r_{kh}(\dot{\partial}_p \Gamma^{*p}_{rl}) - H^p_{rh}(\dot{\partial}_p \Gamma^{*r}_{kl}) - H^p_{kr}(\dot{\partial}_p \Gamma^{*r}_{hl})$$

$$- H^p_{rkh} P^r_{p\ell}\}_{|m} + H^r_{kh|\ell}(\dot{\partial}_p \Gamma^{*p}_{rm}) - H^p_{rh|\ell}(\dot{\partial}_p \Gamma^{*r}_{km}) - H^p_{kr|\ell}(\dot{\partial}_p \Gamma^{*r}_{hm})$$

$$-H^p_{kh|r}(\dot{\partial}_p \Gamma^{*r}_{\ell m}) - H^p_{rkh|\ell} P^r_{Pm} - H^s_{kh}(\dot{\partial}_r \Gamma^{*p}_{s\ell}) P^r_{pm} + H^p_{sh}(\dot{\partial}_r \Gamma^{*s}_{k\ell}) P^r_{pm}$$

$$+ H^p_{ks}(\dot{\partial}_r \Gamma^{*s}_{h\ell}) P^r_{pm} + H^p_{skh} P^s_{r\ell} P^r_{pm} = \lambda_\ell (H_{hk} - H_{kh})_{|m} +$$

$$b_{\ell m}(H_{hk} - H_{kh}) + (\dot{\partial}_p \lambda_\ell) H^p_{kh|m} + \lambda_\ell H^r_{kh}(\dot{\partial}_p \Gamma^{*p}_{rm}) - \lambda_\ell H^p_{rh}(\dot{\partial}_p \Gamma^{*r}_{km})$$

$$-\lambda_\ell H^p_{kr}(\dot{\partial}_p \Gamma^{*r}_{hm}) - \lambda_\ell H^p_{rkh} P^r_{pm} + (\dot{\partial}_p b_{\ell m}) H^p_{kh}.$$

This shows that

$$(H_{hk} - H_{kh})_{|\ell|m} = \lambda_\ell (H_{hk} - H_{kh})_{|m} + b_{\ell m}(H_{hk} - H_{kh})$$

if and only if

(2.12) $$\{H^r_{kh}(\dot{\partial}_p \Gamma^{*p}_{rl}) - H^p_{rh}(\dot{\partial}_p \Gamma^{*r}_{kl}) - H^p_{kr}(\dot{\partial}_p \Gamma^{*r}_{hl}) - H^p_{rkh} P^r_{p\ell}\}_{|m}$$

$$+H^r_{kh|\ell}(\dot{\partial}_p \Gamma^{*p}_{rm}) - H^p_{rh|\ell}(\dot{\partial}_p \Gamma^{*r}_{km}) - H^p_{kr|\ell}(\dot{\partial}_p \Gamma^{*r}_{hm}) - H^p_{kh|r}(\dot{\partial}_p \Gamma^{*r}_{\ell m})$$

$$-H^p_{rkh|\ell} P^r_{Pm} - H^s_{kh}(\dot{\partial}_r \Gamma^{*p}_{s\ell}) P^r_{pm} + H^p_{sh}(\dot{\partial}_r \Gamma^{*s}_{k\ell}) P^r_{pm} + H^p_{ks}(\dot{\partial}_r \Gamma^{*s}_{h\ell}) P^r_{pm}$$

$$+ H^p_{skh} P^s_{r\ell} P^r_{pm}\} = b_{\ell m}(H_{hk} - H_{kh}) + \{(\dot{\partial}_p \lambda_\ell) H^p_{kh|m} + \lambda_\ell H^r_{kh}(\dot{\partial}_p \Gamma^{*p}_{rm})$$

$$-\lambda_\ell H^p_{kr}(\dot{\partial}_p \Gamma^{*r}_{hm}) - \lambda_\ell H^p_{rkh} P^r_{pm} + (\dot{\partial}_p b_{\ell m}) H^p_{kh}.$$

Thus, we conclude

Theorem 2.5. *In K^h–GBR–F_n, the tensor $(H_{hk} - H_{kh})$ is generalized birecurrent if and only if (2.12) holds good.*

Differentiating (1.18) covariantly with respect to x^m in the sense of Cartan and using (1.1a), we get

(2.13) $$K^i_{jkh|\ell|m} + K^i_{j\ell k|h|m} + K^i_{jh\ell|k|m} + y^r\{(\dot{\partial}_s \Gamma^{*i}_{jk}) K^s_{rh\ell|m} +$$

$$(\dot{\partial}_s \Gamma^{*i}_{j\ell}) K^s_{rkh|m} + (\dot{\partial}_s \Gamma^{*i}_{jh}) K^s_{r\ell k|m}\} + y^r\{(\dot{\partial}_s \Gamma^{*i}_{jk})_{|m} K^s_{rh\ell}$$

$$+\left(\dot{\partial}_s\, \Gamma_{j\ell}^{*i}\right)_{|m} K_{rkh}^{s} + \left(\dot{\partial}_s\, \Gamma_{jh}^{*i}\right)_{|m} K_{r\ell k}^{s}\Big\} = 0.$$

Using (2.1) in (2.13), we get

(2.14) $$\lambda_\ell K_{jkh|m}^{i} + \lambda_h K_{j\ell k|m}^{i} + \lambda_k K_{jh\ell|m}^{i} + b_{\ell m} K_{jkh}^{i} + b_{hm} K_{j\ell k}^{i} + b_{km} K_{jh\ell}^{i}$$

$$+y^r\{(\dot{\partial}_s\, \Gamma_{jk}^{*i}) K_{rh\ell|m}^{s} + (\dot{\partial}_s\, \Gamma_{j\ell}^{*i}) K_{rkh|m}^{s} + (\dot{\partial}_s\, \Gamma_{jh}^{*i}) K_{r\ell k|m}^{s}\}$$

$$+y^r\left\{\left(\dot{\partial}_s\, \Gamma_{jk}^{*i}\right)_{|m} K_{rh\ell}^{s} + \left(\dot{\partial}_s\, \Gamma_{j\ell}^{*i}\right)_{|m} K_{rkh}^{s} + \left(\dot{\partial}_s\, \Gamma_{jh}^{*i}\right)_{|m} K_{r\ell k}^{s}\right\} = 0.$$

If Cartan's fourth curvature tensor K_{jkh}^{i} is recurrent which is given by (1.23), (2.14) becomes

(2.15) $$\lambda_\ell\, \lambda_m K_{jkh}^{i} + \lambda_h\, \lambda_m K_{j\ell k}^{i} + \lambda_k\, \lambda_m K_{jh\ell}^{i} + b_{\ell m} K_{jkh}^{i} + b_{hm} K_{j\ell k}^{i} + b_{km} K_{jh\ell}^{i}$$

$$+ \lambda_m\, y^r\{(\dot{\partial}_s\, \Gamma_{jk}^{*i}) K_{rh\ell}^{s} + (\dot{\partial}_s\, \Gamma_{j\ell}^{*i}) K_{rkh}^{s} + (\dot{\partial}_s\, \Gamma_{jh}^{*i}) K_{r\ell k}^{s}\}$$

$$+ y^r\left\{\left(\dot{\partial}_s\, \Gamma_{jk}^{*i}\right)_{|m} K_{rh\ell}^{s} + \left(\dot{\partial}_s\, \Gamma_{j\ell}^{*i}\right)_{|m} K_{rkh}^{s} + \left(\dot{\partial}_s\, \Gamma_{jh}^{*i}\right)_{|m} K_{r\ell k}^{s}\right\} = 0$$

Putting (1.18) in (2.15), we get K_{jkh}^{i}

$$\lambda_\ell\, \lambda_m K_{jkh}^{i} + \lambda_h\, \lambda_m K_{j\ell k}^{i} + \lambda_k\, \lambda_m K_{jh\ell}^{i} + b_{\ell m} K_{jkh}^{i}$$

$$+ b_{hm} K_{j\ell k}^{i} + b_{km} K_{jh\ell}^{i} - \lambda_m(K_{jkh|\ell}^{i} + K_{j\ell k|h}^{i} + K_{jh\ell|k}^{i})$$

$$+ y^r\left\{\left(\dot{\partial}_s\, \Gamma_{jk}^{*i}\right)_{|m} K_{rhl}^{s} + \left(\dot{\partial}_s\, \Gamma_{jl}^{*i}\right)_{|m} K_{rkh}^{s} + \left(\dot{\partial}_s\, \Gamma_{jh}^{*i}\right)_{|m} K_{rlk}^{s}\right\} = 0$$

which can be written as

(2.16) $$b_{\ell m} K_{jkh}^{i} + b_{hm} K_{j\ell k}^{i} + b_{km} K_{jh\ell}^{i} + y^r\left\{\left(\dot{\partial}_s\, \Gamma_{jk}^{*i}\right)_{|m} K_{rh\ell}^{s} +\right.$$

$$\left. +\left(\dot{\partial}_s\, \Gamma_{j\ell}^{*i}\right)_{|m} K_{rkh}^{s} + \left(\dot{\partial}_s\, \Gamma_{jh}^{*i}\right)_{|m} K_{r\ell k}^{s}\right\} = 0.$$

Transvecting (2.16) by y^j, using (1.1a), (1.6) and (1.2b), we get

$$b_{\ell m}\, H_{kh}^{i} + b_{hm}\, H_{\ell k}^{i} + b_{km}\, H_{h\ell}^{i} + P_{sk|m}^{i}\, H_{h\ell}^{s} + P_{s\ell|m}^{i}\, H_{kh}^{s} + P_{sh|m}^{i}\, H_{\ell k}^{s} = 0\,.$$

3. K^h– Generalized Birecurrent Affinely Connected Space

Let us consider an affinely connected or Berwald's space which is characterized by any one of the equivalent conditions (1.19), (1.20), (1.21) and (1.22).

Definition 3.1. The K^h– generalized birecurrent space is called K^h– *generalized birecurrent affinely connected* if it satisfies any one of the conditions (1.19), (1.20), (1.21) and (1.22) and denoted briefly by $K^h – GBR$–*affinely connected space.*

Let us consider $K^h - GBR -$ affinely connected space.

If $\dot{\partial}_j \lambda_\ell = 0$, $\dot{\partial}_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.5) reduces to

(1.3) $$H^i_{jkh|\ell|m} = \lambda_\ell H^i_{jkh|m} + b_{\ell m} H^i_{jkh} .$$

Thus, we conclude

Theorem 3.1. *In $K^h – GBR$–affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then Berwald curvature tensor H^i_{jkh} is generalized bireurrent.*

If $\dot{\partial}_j \lambda_\ell = 0$, $\dot{\partial}_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.7) reduces to

$$H_{jpkh|\ell|m} = \lambda_\ell H_{jpkh|m} + b_{\ell m} H_{jpkh}.$$

Thus, we conclude

Theorem 3.2. *In $K^h – GBR$–affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the associate tensor H_{jpkh} of Berwald curvature tensor H^i_{jkh} is generalized bireurrent.*

If $\dot{\partial}_j \lambda_\ell = 0$, $\dot{\partial}_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.9) reduces to

$$H_{jk|\ell|m} = \lambda_\ell H_{jk|m} + b_{\ell m} H_{jk}.$$

Thus, we conclude

Theorem 3.3. *In $K^h – GBR$–affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the Ricci tensor H_{jk} in sense of Berwald is generalized bireurrent.*

If $\dot{\partial}_j \lambda_\ell = 0$, $\dot{\partial}_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.11) reduces to

$$(H_{hk} - H_{kh})_{|\ell|m} = \lambda_\ell (H_{hk} - H_{kh})_{|m} + b_{\ell m} (H_{hk} - H_{kh}).$$

Thus, we conclude

Theorem 3.4. *In $K^h - GBR$ –affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the tensor $(H_{hk} - H_{kh})$ is generalized bireurrent.*

Now, transvecting (3.1) by y^j, using (1.1a) and (1.8), we get

(3.2) $$H^i_{kh|\ell|m} = \lambda_\ell H^i_{kh|m} + b_{\ell m} H^i_{kh} .$$

Transvecting (3.2) by y^k , using (1.1a) and (1.11), we get

(3.3) $$H^i_{h|\ell|m} = \lambda_\ell H^i_{h|m} + b_{\ell m} H^i_h .$$

Contracting the indices i and h in (3.2) and using (1.13), we get

$$H_{k|\ell|m} = \lambda_\ell H_{k|m} + b_{\ell m} H_k$$

Contracting the indices i and h in (3.3) and using (1.14), we get

$$H_{|\ell|m} = \lambda_\ell H_{|m} + b_{\ell m} H.$$

Transvecting (3.2) by g_{ip} , using (1.1b) and (1.17), we get

$$H_{kp.h|\ell|m} = \lambda_\ell H_{kp.h|m} + b_{\ell m} H_{kp.h}.$$

Thus, we conclude

Theorem 3.5. *In $K^h - GBR$ –affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the $h(v)$– torsion tensor H^i_{kh} , the deviation tensor H^i_h , the curvature vector H_k, the curvature scalar H and the tensor $H_{kp.h}$ are all generalized birecurrent.*

In view of (1.22), the equation (2.14) can be written as

(3.4) $$\lambda_\ell K^i_{jkh|m} + \lambda_h K^i_{j\ell k|m} + \lambda_k K^i_{jh\ell|m} + b_{\ell m} K^i_{jkh} + b_{hm} K^i_{j\ell k} + b_{km} K^i_{jh\ell} = 0.$$

In view of (1.23), (3.4) reduces to

$$(\lambda_\ell \lambda_m + b_{lm}) K^i_{jkh} + (\lambda_h \lambda_m + b_{hm}) K^i_{j\ell k} + (\lambda_k \lambda_m + b_{km}) K^i_{jh\ell} = 0$$

which can be written as

(3.5) $$a_{\ell m} K^i_{jkh} + a_{hm} K^i_{j\ell k} + a_{km} K^i_{jh\ell} = 0 ,$$

where $a_{rm} = \lambda_r \lambda_m + b_{rm}$ is covariant tensor field of second order.

Transvecting (3.5) by y^j, using (1.1a) and (1.6), we get

(3.6) $$a_{\ell m}\, H^i_{kh} + a_{hm}\, H^i_{\ell k} + a_{km}\, H^i_{h\ell} = 0.$$

Thus, we conclude

Theorem 3.6. *In K^h– GBR– affinely connected space, the identities (3.5) and (3.6) are hold good.*

Contracting the indices i and ℓ in (3.5), using (1.3) and (1.5), we get

(3.7) $$a_{pm}\, K^p_{jkh} - a_{hm}\, K_{jk} + a_{km} K_{jh} = 0$$

which can be written as

(3.8) $$K^p_{jkh} = \frac{(a_{hm}\, K_{jk} - a_{km}\, K_{jh})}{a_{pm}}.$$

Thus, we conclude

Theorem 3.7. *In K^h– GBR–affinely connected space, Cartan's fourth curvature tensor K^p_{jkh} is defined by (3.8).*

In view of (1.2b) and (1.22), (1.7) reduces to

(3.9) $$H^{\ i}_{jkh} = K^{\ i}_{jkh}.$$

Putting (3.9) in (3.5), we get

(3.10) $$a_{\ell m}\, H^i_{jkh} + a_{hm}\, H^i_{j\ell k} + a_{km}\, H^i_{jh\ell} = 0.$$

Thus, we conclude

Theorem 3.8. *In $K^h - GBR -$ affinely connected space, Berwald curvature tensor $H^{\ i}_{jkh}$ coincide with Cartan's fourth curvature tensor $K^{\ i}_{jkh}$ and the identity (3.10) holds good.*

Contracting the indices i and ℓ in (3.10), using (1.12), (1.10b) and the skew-symmetric property of Berwald curvature tensor $H^{\ i}_{jkh}$ in it's last two lower indices, we get

$$a_{pm}\, H^p_{jkh} - a_{hm}\, H_{jk} + a_{km}\, H_{jh} = 0.$$

which can be written as

(3.11) $$H^p_{jkh} = \frac{(a_{hm}\, H_{jk} - a_{km}\, H_{jh})}{a_{pm}}.$$

Thus, we conclude

Theorem 3.9. *In* K^h-GBR*–affinely connected space, Berwald curvature tensor is defined by (3.11).*

Contracting the indices i and h in (3.9), using (1.12) and (1.5), we get

$$H_{jk} = K_{jk}.$$

Thus, we conclude

Theorem 3.10. *In* K^h-GBR*–affinely connected space, Ricci tensor* H_{jk} *in sense of Berwald* coincide with *Ricci tensor* K_{jk} *of Cartan's fourth curvature.*

Transvecting (3.9) by g_{ip} , using (1.1b), (1.15) and (1.4), we get

$$H_{jpkh} = K_{jpkh}.$$

Thus, we conclude

Theorem 3.11. *In* K^h-GBR*–affinely connected space, the associate curvature tensor* H_{jpkh} *of Berwald curvature tensor coincide with the associate curvature tensor* K_{jpkh} *of Cartan's fourth curvature tensor.*

4. Conclusion

(3.1) The K^h- generalized birecurrent space is called K^h- generalized birecurrent affinely connected if it satisfies any one of the conditions (1.19), (1.20), (1.21) and (1.22).

(3.2) In K^h-GBR–affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then Berwald curvature tensor H^i_{jkh} is generalized bireurrent.

(3.3) In $K^h - GBR -$affinely connected space, if the directional derivative of covariant vector field a and covariant tensor of second order are vanish, then the $h(v)-$ torsion tensor H^i_{kh} , the deviation tensor H^i_h , the curvature vector H_k, the curvature scalar H and the tensor $H_{kp.h}$ are all generalized birecurrent.

(3.4) In K^h-GBR–affinely connected space, Cartan's fourth curvature tensor K^p_{jkh} is defined by (3.8).

(3.5) In K^h-GBR–affinely connected space, Berwald curvature tensor $H^{\ i}_{jkh}$ coincide with Cartan's fourth curvature tensor $K^{\ i}_{jkh}$ and the identity (3.10) holds good.

(3.6) In K^h-GBR–affinely connected space, Ricci tensor H_{jk} in sense of Berwald coincide with Ricci tensor K_{jk} of Cartan's fourth curvature.

(3.7) In K^h- GBR−affinely connected space, the associate curvature tensor H_{jpkh} of Berwald curvature tensor coincide with the associate curvature tensor K_{jpkh} of Cartan's fourth curvature tensor.

5. Recommendations

Authors recommend the need for the continuing research and development in affinely connected space and its relation with other spaces.

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