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# A Positivity-Preserving Nonstandard Finite Difference Scheme for Parabolic System with Cross-Diffusion Equations and Nonlocal Initial Conditions

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**Abstract**

An important number of ecological phenomena can be modeled using nonlinear diffusion partial differential equations. This paper considers a system of cross-diffusion equations with nonlocal initial conditions. Such equations arise as steady-state equations in an age-structured predator-prey model with diffusion. We use the nonstandard finite difference method developed by Mickens. These types of schemes are made by the following two rules: first, renormalization of step size for the denominator function of representations of derivatives, and secondly, nonlocal representations of nonlinear terms. We obtained a scheme that preserves the positivity of solutions. Furthermore, this scheme is explicit and functional relationship is obtained between time, space, and age step sizes.

***Keywords*****:** Cross-diffusion equations; finite difference methods; nonlocal initial conditions; nonstandard finite difference schemes; positivity of solutions; predator-prey model.

## 1. Introduction

The application of partial differential equations is very common in the natural and engineering sciences. In particular, the nonlinear modeling of reaction and/or diffusion phenomena [1]. In this paper we consider the equations for studying an age-structure of the predator-prey model, defined by the following parabolic equations with nonlocal initial conditions. The predator population is putting pressure on the prey and all populations suffer fluctuations related to space [2].

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If $u = u(t,x,a) \geq 0$ and $v = v(t,x,a) \geq 0$ are respectively the densities of populations of predator and prey that live in the same spatial region $\Omega$ and are structured by age $a \in [0, a_m)$ for a maximal age $a_m > 0$, and spatial position $x \in \Omega$, then the model will be [2] :

$$\begin{cases} \partial_t u + \partial_a u - \Delta_x\big((\delta_1 + \gamma v)u\big) = -\alpha_1 u^2 - \alpha_2 uv, t \geq 0, a \in (0, a_m), x \in \Omega \\ \partial_t v + \partial_a v - \delta_2 \Delta_x v = -\beta_1 v^2 + \beta_2 uv, t \geq 0, a \in (0, a_m), x \in \Omega \end{cases} \quad (1)$$

These equations are subject to the constraints

$$\begin{cases} u(t,0,x) = \int_0^{a_m} B_1(a) u(t,a,x) da, t \geq 0, x \in \Omega \\ v(t,0,x) = \int_0^{a_m} B_2(a) v(t,a,x) da, t \geq 0, x \in \Omega \end{cases} \quad (2)$$

and the conditions to space limitations:

$$\begin{cases} u(t,x,a) = 0, t \geq 0, a \in (0, a_m), x \in \partial\Omega \\ v(t,x,a) = 0, t \geq 0, a \in (0, a_m), x \in \partial\Omega \end{cases} \quad (3)$$

and also the additional initial conditions.

The Laplacian $\Delta_x$ indicates the spatial movement of the prey population, $\delta_1 > 0$ is the intrinsic dispersion coefficient, which reflects the growing strength of dispersion prey because of interference with the growth of the predator population and $\gamma \geq 0$ is the pressure coefficient of the population of predators. For notational simplicity, we will take $\delta_1 = \delta_2 = 1$. The right hand sides of the system (1) taking into account the interactions within and outside of two specific populations with coefficient $\alpha_1, \alpha_2, \beta_1$ and $\beta_2 > 0$. Equations (3) describe the rates of creation of new individuals to $B_1$ and $B_2$.

The objective of this paper is to construct a finite difference scheme that preserves the positivity of solutions. The method used in this construction is based on nonstandard discretisation technique created by Ronald Mickens. In particular, the scheme must be explicit and requires a functional relationship between time, space and age step sizes. In general, the absence of this restriction leads to the existence of numerical instabilities, i.e. the solutions to the difference equations do not correspond to the solutions of the continuous equations [2].

The remainder of this paper is organized as follows: Section 2 gives a brief overview of the philosophy of construction of nonstandard finite difference scheme (NSFDS), while Section 3 presents the construction of a new NSFDS. Finally, Section 4 provides a summary discussion results.

## 2. Nonstandard modeling

The procedures for constructing NSFDS are stated in [3]. These types of schemes are made by the following two rules [4,5]: first, how to determine the discrete representations of derivatives, and secondly, what is the form of nonlinear terms.

The first concept involves the generalization of the representations of derivatives form [2,3]:

$$\frac{dx}{dt} \rightarrow \frac{x_{n+1} - x_n}{\phi(h, \lambda)} \tag{4}$$

where $t_n = (\Delta t)n = nh, x_n$ is the approximation of $x(t_n)$, and the denominator function satisfies :

$$\phi(h, \lambda) = h + \vartheta(h^2) \tag{5}$$

In equations (4) and (5), $\lambda$ represents the parameters of the differential equation. This construction technique can easily be extended to partial differential equations.

The second concept is related to the modeling of nonlinear terms, for example, $x^2$. In general, the nonlinear terms are replaced by nonlocal representation [4]:

$$x^2 \rightarrow \begin{cases} 2x_n^2 - x_{n+1}x_n \\ x_{n+1}x_n \end{cases} \tag{6}$$

Both concepts are discussed in detail in [3]. A good example is the advection-reaction equation [4],[6]

$$u_t + u_x = u(1 - u) \tag{7}$$

where the exact numerical scheme is

$$\frac{u_m^{n+1} - u_m^n}{\phi(\Delta t)} + \frac{u_m^n - u_{m-1}^n}{\phi(\Delta x)} = u_{m-1}^n(1 - u_m^{n+1}) \tag{8}$$

The denominator function is

$$\phi(z) = e^z - 1 \tag{9}$$

$t_n = (\Delta t)n, x_m = (\Delta t)m$ and $u_m^n$ is the representation of $u(x_m, t_n)$. The major difficulty in the numerical simulation of partial differential equations of which we do not know the exact solution often leads to consider

$$\phi(z) = z \tag{10}$$

### 3. Nonstandard finite difference scheme

In one dimension of space, the system (1) can be written as

$$\begin{cases} \partial_t u + \partial_a u = \delta_1 \partial_{xx} u + \gamma u \partial_{xx} v + \gamma v \partial_{xx} u + 2\gamma \partial_x u \partial_x v - \alpha_1 u^2 - \alpha_2 uv, \\ \quad t \geq 0, a \in (0, a_m)\ , x \in \Omega \\ \partial_t v + \partial_a v = \delta_2 \partial_{xx} v - \beta_1 v^2 + \beta_2 uv, t \geq 0, a \in (0, a_m)\ , x \in \Omega \end{cases} \tag{11}$$

We now suggest the following scheme for system (11):

$$\begin{cases} \dfrac{u_{m,p}^{n+1} - u_{m,p}^n}{\Delta t} + \dfrac{u_{m,p}^n - u_{m-1,p}^n}{\Delta a} = \delta_1 \dfrac{u_{m,p+1}^n - 2u_{m,p}^n + u_{m,p-1}^n}{(\Delta x)^2} \\ +\gamma u_{m,p}^n \dfrac{v_{m,p+1}^n - 2v_{m,p}^n + v_{m,p-1}^n}{(\Delta x)^2} + \gamma v_{m,p}^n \dfrac{u_{m,p+1}^n - 2u_{m,p}^n + u_{m,p-1}^n}{(\Delta x)^2} \\ +\gamma \left(\dfrac{u_{m,p}^n - u_{m,p-1}^n}{\Delta x}\right)\left(\dfrac{v_{m,p}^n - v_{m,p-1}^n}{\Delta x}\right) + \gamma \left(\dfrac{u_{m,p+1}^n - u_{m,p}^n}{\Delta x}\right)\left(\dfrac{v_{m,p+1}^n - v_{m,p}^n}{\Delta x}\right) \\ -\alpha_1 u_{m,p}^{n+1} u_{m-1,p}^n - \alpha_2 u_{m,p}^{n+1} v_{m-1,p}^n \\ \dfrac{v_{m,p}^{n+1} - v_{m,p}^n}{\Delta t} + \dfrac{v_{m,p}^n - v_{m-1,p}^n}{\Delta a} = \delta_2 \dfrac{v_{m,p+1}^n - 2v_{m,p}^n + v_{m,p-1}^n}{(\Delta x)^2} \\ -\beta_1 v_{m,p}^{n+1} v_{m-1,p}^n + \beta_2 u_{m-1,p}^n v_{m-1,p}^n \end{cases} \tag{12}$$

where we have the following discrete representations :

$$\begin{cases} \partial_t u \to \dfrac{u_{m,p}^{n+1} - u_{m,p}^n}{\Delta t}\ ;\ \partial_a u \to \dfrac{u_{m,p}^n - u_{m-1,p}^n}{\Delta a} \\ \partial_{xx} u \to \dfrac{u_{m,p+1}^n - 2u_{m,p}^n + u_{m,p-1}^n}{(\Delta x)^2}\ ;\ \partial_{xx} v \to \dfrac{v_{m,p+1}^n - 2v_{m,p}^n + v_{m,p-1}^n}{(\Delta x)^2} \\ 2\gamma \partial_x u \partial_x v \to \gamma \left(\dfrac{u_{m,p}^n - u_{m,p-1}^n}{\Delta x}\right)\left(\dfrac{v_{m,p}^n - v_{m,p-1}^n}{\Delta x}\right) \\ +\gamma \left(\dfrac{u_{m,p+1}^n - u_{m,p}^n}{\Delta x}\right)\left(\dfrac{v_{m,p+1}^n - v_{m,p}^n}{\Delta x}\right) \\ -u^2 \to -u_{m,p}^{n+1} \cdot u_{m-1,p}^n\ ;\ -uv \to -u_{m,p}^{n+1} \cdot v_{m-1,p}^n\ ;\ uv \to u_{m-1,p}^n \cdot v_{m-1,p}^n \end{cases} \tag{13}$$

Solving $u_{m,p}^{n+1}$ and $v_{m,p}^{n+1}$ in (12) we get

$$\begin{cases} u_{m,p}^{n+1} = \dfrac{u_{m,p}^n\left(1 - R_1 - 2\delta_1 R_2 - 2\gamma R_2 v_{m,p}^n\right) + R_1 u_{m-1,p}^n + \delta_1 R_2 \left(u_{m,p+1}^n + u_{m,p-1}^n\right) + \gamma R_2 \left(u_{m,p-1}^n v_{m,p-1}^n + u_{m,p+1}^n v_{m,p+1}^n\right)}{1 + \alpha_1 \Delta t u_{m-1,p}^n + \alpha_2 \Delta t v_{m-1,p}^n} \\ v_{m,p}^{n+1} = \dfrac{v_{m,p}^n\left(1 - R_1 - 2\delta_2 R_2\right) + R_1 v_{m-1,p}^n + \delta_2 R_2 \left(v_{m,p+1}^n + v_{m,p-1}^n\right) + \beta_2 \Delta t u_{m-1,p}^n v_{m-1,p}^n}{1 + \beta_1 \Delta t v_{m-1,p}^n} \end{cases} \tag{14}$$

where

$$R_1 = \frac{\Delta t}{\Delta a} \text{ and } R_2 = \frac{\Delta t}{(\Delta x)^2} \tag{15}$$

Positivity of solutions provides that

$$u_{m,p}^n \geq 0\ , v_{m,p}^n \geq 0 \Rightarrow u_{m,p}^{n+1} \geq 0, v_{m,p}^{n+1} \geq 0 \tag{16}$$

which leads to the following conditions

$$\begin{cases} 1 - R_1 - 2\delta_1 R_2 - 2\gamma R_2 v_{m,p}^n \geq 0 \\ 1 - R_1 - 2\delta_2 R_2 \geq 0 \end{cases} \quad (17)$$

and gives

$$\begin{cases} \Delta t \leq \dfrac{(\Delta a)(\Delta x)^2}{(\Delta x)^2 + 2\delta_1(\Delta a) + 2\gamma R_2 v_{m,p}^n} \\ \Delta t \leq \dfrac{(\Delta a)(\Delta x)^2}{(\Delta x)^2 + 2\delta_2(\Delta a)} \end{cases} \quad (18)$$

Taking $\delta_1 = \delta_2 = 1$, we find that

$$\Delta t \leq \frac{(\Delta a)(\Delta x)^2}{(\Delta x)^2 + 2(\Delta a) + 2\gamma R_2 v_{m,p}^n} \quad (19)$$

In this paper, we only consider the equality in equation (19) and system (14) becomes :

$$\begin{cases} u_{m,p}^{n+1} = \dfrac{R_1 u_{m-1,p}^n + R_2\left(u_{m,p+1}^n + u_{m,p-1}^n\right) + \gamma R_2\left(u_{m,p-1}^n v_{m,p-1}^n + u_{m,p+1}^n v_{m,p+1}^n\right)}{1 + \alpha_1 \Delta t u_{m-1,p}^n + \alpha_2 \Delta t v_{m-1,p}^n} \\ v_{m,p}^{n+1} = \dfrac{v_{m,p}^n(1 - R_1 - 2\delta_2 R_2) + R_1 v_{m-1,p}^n + R_2\left(v_{m,p+1}^n + v_{m,p-1}^n\right) + \beta_2 \Delta t u_{m-1,p}^n v_{m-1,p}^n}{1 + \beta_1 \Delta t v_{m-1,p}^n} \end{cases} \quad (20)$$

Note that the time step size in (19) is not constant. It is also clear that (20) preserves the positivity of solutions. For simulation, select $\Delta a, \Delta x$ and then calculate $\Delta t$ ,$R_1$ and $R_2$ at every step.

#### 4. Discussion

If $\gamma = 0$ in (11), then we find a system of reaction-diffusion equations with nonlocal initial conditions [7]

$$\begin{cases} \partial_t u + \partial_a u = \delta_1 \partial_{xx} u - \alpha_1 u^2 - \alpha_2 uv, t \geq 0, a \in (0, a_m) \ , x \in \Omega \\ \partial_t v + \partial_a v = \delta_2 \partial_{xx} v - \beta_1 v^2 + \beta_2 uv, t \geq 0, a \in (0, a_m) \ , x \in \Omega \end{cases} \quad (21)$$

The following nonstandard finite difference scheme was constructed for the system (21) in [8]

$$\begin{cases} u_{m,p}^{n+1} = \dfrac{u_{m,p}^n(1 - R_1 - 2\delta_1 R_2) + R_1 u_{m-1,p}^n + \delta_1 R_2\left(u_{m,p+1}^n + u_{m,p-1}^n\right)}{1 + \alpha_1 \Delta t u_{m-1,p}^n + \alpha_2 \Delta t v_{m-1,p}^n} \\ v_{m,p}^{n+1} = \dfrac{v_{m,p}^n(1 - R_1 - 2\delta_2 R_2) + R_1 v_{m-1,p}^n + \delta_2 R_2\left(v_{m,p+1}^n + v_{m,p-1}^n\right) + \beta_2 \Delta t u_{m-1,p}^n v_{m-1,p}^n}{1 + \beta_1 \Delta t v_{m-1,p}^n} \end{cases} \quad (22)$$

This is in fact the situation when $\gamma = 0$ in (14).

If more $\delta_1 = \delta_2 = 1$, then we get

$$\Delta t \leq \frac{(\Delta a)(\Delta x)^2}{(\Delta x)^2 + 2(\Delta a)} \tag{23}$$

And considering equality in equation (23), the scheme (22) becomes:

$$\begin{cases} u_{m,p}^{n+1} = \dfrac{R_1 u_{m-1,p}^n + R_2\left(u_{m,p+1}^n + u_{m,p-1}^n\right)}{1 + \alpha_1 \Delta t u_{m-1,p}^n + \alpha_2 \Delta t v_{m-1,p}^n} \\ v_{m,p}^{n+1} = \dfrac{R_1 v_{m-1,p}^n + R_2\left(v_{m,p+1}^n + v_{m,p-1}^n\right) + \beta_2 \Delta t u_{m-1,p}^n v_{m-1,p}^n}{1 + \beta_1 \Delta t v_{m-1,p}^n} \end{cases} \tag{24}$$

where

$$R_1 = \frac{(\Delta x)^2}{(\Delta x)^2 + 2(\Delta a)}, \; R_2 = \frac{(\Delta a)}{(\Delta x)^2 + 2(\Delta a)} \text{ and } 2R_2 + R_1 = 1 \tag{25}$$

**5. Conclusion**

We have constructed a nonstandard finite difference scheme for a system of nonlinear parabolic cross-diffusion equations with nonlocal initial conditions. This scheme is made by using non local representations of nonlinear terms, and preserves the positivity of solutions. Furthermore, this scheme is explicit and we obtained a variable time step size which depends on space and age step sizes.

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