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On a Generalized βR − Birecurrent Finsler Space

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Abstract

In the present paper, we introduced a Finsler space whose Cartan's third curvature tensor R^i_{jkh} satisfies $\mathcal{B}_m\mathcal{B}_n R^i_{jkh} = a_{mn}R^i_{jkh} + b_{mn}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) - 2y^r\mu_n\mathcal{B}_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm})$. where a_{mn} and b_{mn} are non-zero covariant tensor fields of second order called recurrence tensor fields, such space is called as a generalized βR −birecurrent Finsler space.

The curvature tensor H^i_{jkh} , the torsion tensor H^i_{kh} ,the deviation tensor H^i_h, the Ricci tensors (H_{jk}, R_{jk}), the vector H_k and the scalar curvature tensor H of such space are non-vanishing. Under certain conditions, a generalized βR −birecurrent Finsler space becomes Landsberg space . Some conditions have been pointed out which reduce a generalized βR −birecurrent Finsler space $F_n (n > 2)$ into Finsler space of scalar curvature.

Keywords: Finsler space; Generalized βR −birecurrent Finsler space; Ricci tensor; Landsberg space; Finsler space of scalar curvature.

1. Introduction

H.S. Ruse[4] considered a three dimensional Riemannian space having the recurrent of curvature tensor and he called such space as Riemannian space of recurrent curvature. This idea was extended to n-dimensional Riemannian and non- Riemannian space by A.G. Walker [1],Y.C. Worg [14] ,Y.C. Worg and K. Yano [15] and others.

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S. Dikshit [13] introduced a Finsler space whose Berwald curvature tensor H^i_{jkh} satisfies recurrence property in the sense of Berwald, F.Y.A.Qasem and A.A.M.Saleem [3] discussed general Finsler space for the hv −curvature tensor U^i_{jkh} satisfies the birecurrence property with respect to Berwald's coefficient G^i_{jk} and they called it *UBR- Finsler space*. P.N.pandey, S.Saxena and A.Goswami [8] introduced a Finsler space whose Berwald curvature tensor H^i_{jkh} satisfies generalized recurrence property in the sense of Berwald they called such space generalized H-recurrent Finsler space .

(1.1) a) $y_i\, y^i = F^2$ b) $g_{ij} = \dot\partial_i y_j = \dot\partial_j y_i$ c) $\mathcal{B}_k\, y^i = 0$

d) $C_{ijk}\, y^i = C_{kij}\, y^i = C_{jki}\, y^i = 0$ e) $\mathcal{B}_k\, g_{ij} = -2C_{ijkh}y^h = -\,2y^h\mathcal{B}_h C_{ijk}$

f) $G^i_{jkh}\, y^j = G^i_{hjk}\, y^j = G^i_{khj} y^j = 0$.

The unit vector l^i and the associate vector l_i is defined by

(1.2) $a)$ $l^i = \frac{y^i}{F}$ $b)$ $l_i = g_{ij} l^j = \dot\partial_i F = \frac{y_i}{F}$.

The processes of Berwald's covariant differentiation and the partial differentiation commute according to

(1.3) $(\dot\partial_k \mathcal{B}_h - \mathcal{B}_k \dot\partial_h)\, T^i_j = T^r_j G^i_{khr} - T^i_r G^r_{khj}$.

The tensor H^i_{jkh} satisfies the relation

(1.4) $H^i_{jkh}\, y^j = H^i_{kh}$.

(1.5) $H^i_{jkh} = \dot\partial_j H^i_{kh}$.

The torsion tensor H^i_{kh} satisfies

(1.6) $H^i_{kh}\, y^k = H^i_h$,

(1.7) $R^i_{jkh}\, y^j = H^i_{kh}$,

(1.8) $H_{jk} = H^i_{jki}$,

(1.9) $H_k = H^i_{ki}$,

and

(1.10) $H = \frac{1}{n-1} H^i_i$.

where H_{jk} and H are called *h-Ricci tensor* [7] and *curvature scalar* respectively. Since contraction of the indices does not affect the homogeneity in y^i , hence the tensors H_{rk} , H_r and the scalar H are also homogeneous of degree zero, one and two in y^i respectively . The above tensors are also connected by

(1.11) $H_{jk}\, y^j = H_k$,

(1.12) $H_{jk} = \dot{\partial}_j H_k$,

(1.13) $H_k\, y^k = (n-1)H$.

(1.14) $H^{\,i}_{kh} = \dot{\partial}_k H^{\,i}_{h}$.

The necessary and sufficient condition for a Finsler space $F_n (n > 2)$ to be a Finsler space of scalar curvature is given by

(1.15) $H^i_h = F^2 R(\,\delta^i_h - l^i\, l_h\,)$.

A Finsler space F_n is said to be Landsberg space if satisfies

(1.16) $y_r G^r_{ijk} = 0$.

The Ricci tensor R_{jk} is given by

(1.17) $R^{\,i}_{jki} = R_{jk}$.

2. Generalized $\boldsymbol{\beta R}$ −Birecurrent Finsler Space

A Finsler space whose Cartan's third curvature tensor R^i_{jkh} satisfies

(2.1) $\mathcal{B}_n R^i_{jkh} = \lambda_n R^i_{jkh} + \mu_n(\delta^i_k g_{jh} - \delta^i_h g_{jk})$, $R^i_{jkh} \neq 0$, where λ_n and μ_n are non-zero covariant vector fields and called the recurrence vector fields, we shall call such Finsler space as a generalized R- recurrent Finsler space.

Differentiating (2.1) covariantly with respect to x^m in the sense of Berwald and using (1.1e) ,we get

(2.2) $$\mathcal{B}_m\mathcal{B}_n R^i_{jkh} = (\,\mathcal{B}_m\lambda_n + \lambda_n\,\lambda_m)R^i_{jkh} + (\,\lambda_n\mu_m + \mathcal{B}_m\mu_n)\big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\big)$$

$$-2y^r\mu_n\mathcal{B}_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}).$$

Which can be written as

(2.3) $$\mathcal{B}_m\mathcal{B}_n R^i_{jkh} = a_{mn} R^i_{jkh} + b_{mn}\big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\big) - 2y^r\mu_n\mathcal{B}_r(\delta^i_k C_{jhm} - \delta^i_h C_{jkm}).$$

where $a_{mn} = \mathcal{B}_m\lambda_n + \lambda_n \lambda_m$ and $b_{mn} = \lambda_n\mu_m + \mathcal{B}_m\mu_n$ are non-zero covariant tensor fields of second order .

Definition 2.1. A Finsler space F_n whose Cartan's third curvature tensor R^i_{jkh} satisfies the condition (2.3) will be called generalized βR-birecurrent Finsler space ,we shall denote it $G\beta R - BR - F_n$.

Transvecting (2.3) by y^j , using (1.1c) , (1.7) and (1.1d), we get

(2.4) $$\mathcal{B}_m\mathcal{B}_n H^i_{kh} = a_{mn}H^i_{kh} + b_{mn}(\delta^i_k y_h - \delta^i_h y_k).$$

Further transvecting (2.4) by y^k , using (1.1c) , (1.6) and (1.1a) , we get

(2.5) $$\mathcal{B}_m\mathcal{B}_n H^i_h = a_{mn}H^i_h + b_{mn}(y^i y_h - \delta^i_h F^2).$$

Thus, we have

Theorem 2.1. In $G\beta R - BR - F_n$, Berwald second covariant derivative of the $h(v)$ −torsion tensor H^i_{kh} and the deviation tensor H^i_h is given by the equations (2.4) and (2.5) ,respectively.

Contracting the indices i and h in (2.3), (2.4) and (2.5) , respectively and using (1.17) ,(1.9) and (1.10) , we get

(2.6) $$\mathcal{B}_m\mathcal{B}_n R_{jk} = a_{mn}R_{jk} + (1-n)b_{mn}g_{jk} - 2(1-n)y^r\mu_n\mathcal{B}_r C_{jkm}.$$

(2.7) $$\mathcal{B}_m\mathcal{B}_n H_k = a_{mn}H_k + (1-n)b_{mn}y_k.$$

and

(2.8) $$\mathcal{B}_m\mathcal{B}_n H = a_{mn}H - b_{mn}F^2.$$

Thus, we have

Theorem 2.2. In $G\beta R - BR - F_n$, the $R -$ Ricci tensor R_{jk} , the curvature vector H_k and the scalar curvature H are non-vanishing .

Differentiating (2.7) partially with respect to y^j and using (1.1b), we get

(2.9) $$\dot{\partial}_j(\mathcal{B}_m\mathcal{B}_n H_k) = (\dot{\partial}_j a_{mn})H_k + a_{mn}(\dot{\partial}_j H_k) + (1-n)(\dot{\partial}_j b_{mn})y_k + (1-n)b_{mn}g_{jk}.$$

Using commutation formula exhibited by (1.3) for $(\mathcal{B}_n H_k)$ in (2.9) and using (1.12), we get

(2.10) $$\mathcal{B}_m\dot{\partial}_j(\mathcal{B}_n H_k) - (\mathcal{B}_r H_k)G^r_{jmn} - (\mathcal{B}_n H_r)G^r_{jmk} = (\dot{\partial}_j a_{mn})H_k$$

$$+a_{mn}H_{jk} + (1-n)(\dot{\partial}_j b_{mn})y_k + (1-n)b_{mn}g_{jk} .$$

Again applying the commutation formula exhibited by (1.3) for (H_k), we get

(2.11) $$\mathcal{B}_m\mathcal{B}_n H_{jk} - \mathcal{B}_m\big(H_r G^r_{knj}\big) - (\mathcal{B}_r H_k)G^r_{jmn} - (\mathcal{B}_n H_r)G^r_{jmk}$$

$$= \big(\dot{\partial}_j a_{mn}\big)H_k + a_{mn}H_{jk} + (1-n)\big(\dot{\partial}_j b_{mn}\big)y_k + (1-n)b_{mn}g_{jk}\ .$$

This shows that

(2.12) $$\mathcal{B}_m\mathcal{B}_n H_{jk} = a_{mn}H_{jk} + (1-n)b_{mn}g_{jk}.$$

if and only if

(2.13) $$-\mathcal{B}_m\big(H_r G^r_{knj}\big) - (\mathcal{B}_r H_k)G^r_{jmn} - (\mathcal{B}_n H_r)G^r_{jmk} = \big(\dot{\partial}_j a_{mn}\big)H_k + (1-n)\big(\dot{\partial}_j b_{mn}\big)y_k.$$

Thus, we have

Theorem 2.3. In $G\beta R - BR - F_n$, the H −Ricci tensor H_{jk} is given by equation (2.12) if and only if (2.13) holds good.

Transvecting (2.11) by y^k , using (1.1c) , (1.1f), (1.11) (1.13) and (1.1a), we get

(2.14) $$\mathcal{B}_m\mathcal{B}_n H_j - (n-1)(\mathcal{B}_r H)G^r_{jmn} = (n-1)\big(\dot{\partial}_j a_{mn}\big)H + a_{mn}H_j + (1-n)\big(\dot{\partial}_j b_{mn}\big)F^2$$

$$+(1-n)b_{mn}y_j.$$

Using (2.7) in (2.14), we get

(2.15) $$(\mathcal{B}_r H)G^r_{jmn} = -\big(\dot{\partial}_j a_{mn}\big)H + \big(\dot{\partial}_j b_{mn}\big)F^2.$$

Suppose $(\mathcal{B}_r H)G^r_{jmn} = 0$, in view of (2.15) , we get

(2.16) $$-\big(\dot{\partial}_j a_{mn}\big)H + \big(\dot{\partial}_j b_{mn}\big)F^2 = 0.$$

Which can be written as

(2.17) $$\big(\dot{\partial}_j b_{mn}\big) = \frac{(\dot{\partial}_j a_{mn})H}{F^2}\ .$$

If the covariant tensor field a_{mn} is independent of y^i, equation (2.17) shows that the covariant tensor field b_{mn} is independent of y^i . conversely , if the covariant tensor b_{mn} is independent of y^i , we get $H\big(\dot{\partial}_j a_{mn}\big) = 0$.

In view theorem2.2 , the condition $H\big(\dot{\partial}_j a_{mn}\big) = 0$ implies $\dot{\partial}_j a_{mn} = 0$,i.e. the covariant tensor field a_{mn} is also independent of y^i . this leads to

Theorem 2.4. In $G\beta R - BR - F_n$ **,** the covariant tensor field b_{mn} is independent of the directional arguments if and only if the covariant tensor field a_{mn} is independent of the directional arguments provided $(\mathcal{B}_r H) G^r_{jmn} = 0$.

Suppose the tensor a_{mn} is not independent of y^i and in view of (2.11) , (2.12) and (2.17), we get

(2.18) $$-\mathcal{B}_m\big(H_r G^r_{knj}\big) - (\mathcal{B}_r H_k) G^r_{jmn} - (\mathcal{B}_n H_r) G^r_{jmk} = \dot{\partial}_j a_{mn}(H_k - \tfrac{(n-1)}{F^2} H y_k).$$

Transvecting (2.18) by y^m , using (1.1c) and (1.1f) , we get

(2.19) $$-\mathcal{B}_m\big(H_r G^r_{knj}\big) y^m = (\,\dot{\partial}_j a_{mn}) y^m (H_k - \tfrac{(n-1)}{F^2} H y_k).$$

Which implies

(2.20) $$-\mathcal{B}_m\big(H_r G^r_{knj}\big) y^m = (\dot{\partial}_j a_n - a_{jn})(H_k - \tfrac{(n-1)}{F^2} H y_k)\,.$$

where $a_{mn} y^m = a_n$

Suppose $\mathcal{B}_m\big(H_r G^r_{knj}\big) y^m = 0$, equation (2.20) has at least one of the following conditions

(2.21) a) $a_{jn} = \dot{\partial}_j a_n$, b) $H_k = \frac{(n-1)}{F^2} H y_k$.

Thus, we have

Theorem 2.5. In $G\beta R - BR - F_n$, which the covariant tensor field a_{mn} is not independent of the directional argument at least one of the conditions(2.21a) and (2.21b) hold.

Suppose (2.21b) holds, then (2.18) implies

(2.22) $$-\mathcal{B}_m\left(\tfrac{(n-1)H}{F^2} y_r G^r_{knj}\right) - \left(\mathcal{B}_r \tfrac{(n-1)H}{F^2} y_k\right) G^r_{jmn} - \left(\mathcal{B}_n \tfrac{(n-1)H}{F^2} y_r\right) G^r_{jmk} = 0.$$

Transvecting (2.22) by y^m and using (1.1c) and (1.1f), we get

(2.23) $$\mathcal{B}_m(H) y_r G^r_{knj} y^m + H\big(\mathcal{B}_m G^r_{knj}\big) y_r y^m = 0.$$

If $H\big(\mathcal{B}_m G^r_{knj}\big) y_r y^m = 0$, the equation (2.23) implies

(2.24) $y_r G^r_{knj} = 0$, since $\mathcal{B}_m(H) y^m \neq 0$

Therefore the space is Landsberg space

Thus , we have

Theorem 2.6. An $G\beta R - BR - F_n$ is Landsberg space if condition (2.21b) holds and provided $H(\mathcal{B}_m G^r_{knj}) y_r y^m = 0$.

If the covariant tensor field $a_{jn} \neq \dot{\partial}_j a_n$, in view of theorem2.5 , (2.21b) holds good. In view of this fact, we may rewrite theorem2.6 in the following

Theorem 2.7. An $G\beta R - BR - F_n$ is necessarily Landsberg space provided

$a_{jn} \neq \dot{\partial}_j a_n$ and $H(\mathcal{B}_m G^r_{knj}) y_r y^m = 0$.

Differentiating (2.4) partially with respect to y^j , using (1.5) and (1.1b), we get

$$\dot{\partial}_j(\mathcal{B}_m\mathcal{B}_n H^i_{kh}) = (\dot{\partial}_j a_{mn}) H^i_{kh} + a_{mn} H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k) + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) . \tag{2.25}$$

Using commutation formula exhibited by (1.3) for $(\mathcal{B}_n H^i_{kh})$ in (2.25) , we get

$$\mathcal{B}_m(\dot{\partial}_j \mathcal{B}_n H^i_{kh}) - (\mathcal{B}_r H^i_{kh}) G^r_{jmn} + (\mathcal{B}_n H^r_{kh}) G^i_{jmr} - (\mathcal{B}_n H^i_{rk}) G^r_{jmh} - (\mathcal{B}_n H^i_{hr}) G^r_{jmk} = (\dot{\partial}_j a_{mn}) H^i_{kh} + a_{mn} H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k) + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}). \tag{2.26}$$

Again applying the commutation formula exhibited by (1.3) for (H^i_{kh}) in (2.26) and using (1.5), we get

$$\mathcal{B}_m(\mathcal{B}_n H^i_{jkh} + H^r_{kh} G^i_{jnr} - H^i_{rk} G^r_{jnh} - H^i_{hr} G^r_{jnk}) - (\mathcal{B}_r H^i_{kh}) G^r_{jmn} + (\mathcal{B}_n H^r_{kh}) G^i_{jmr} - (\mathcal{B}_n H^i_{rk}) G^r_{jmh} - (\mathcal{B}_n H^i_{hr}) G^r_{jmk} = (\dot{\partial}_j a_{mn}) H^i_{kh} + a_{mn} H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k) + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}).$$

Above equation can be written as

$$\mathcal{B}_m\mathcal{B}_n H^i_{jkh} + (\mathcal{B}_m H^r_{kh}) G^i_{jnr} + H^r_{kh}(\mathcal{B}_m G^i_{jnr}) - (\mathcal{B}_m H^i_{rk}) G^r_{jnh} - H^i_{rk}(\mathcal{B}_m G^r_{jnh}) - (\mathcal{B}_m H^i_{hr}) G^r_{jnk} - H^i_{hr}(\mathcal{B}_m G^r_{jnk}) - (\mathcal{B}_r H^i_{kh}) G^r_{jmn} + (\mathcal{B}_n H^r_{kh}) G^i_{jmr} - (\mathcal{B}_n H^i_{rk}) G^r_{jmh} - (\mathcal{B}_n H^i_{hr}) G^r_{jmk} = (\dot{\partial}_j a_{mn}) H^i_{kh} + a_{mn} H^i_{jkh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k) + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) . \tag{2.27}$$

This shows that

(2.28) $\mathcal{B}_m\mathcal{B}_n H^i_{jkh} = a_{mn}H^i_{jkh} + b_{mn}(\delta^i_k g_{jh} - \delta^i_h g_{jk})$.

if and only if

(2.29) $(\mathcal{B}_m H^r_{kh})G^i_{jnr} + H^r_{kh}(\mathcal{B}_m G^i_{jnr}) - (\mathcal{B}_m H^i_{rk})G^r_{jnh} - H^i_{rk}(\mathcal{B}_m G^r_{jnh})$

$-(\mathcal{B}_m H^i_{hr})G^r_{jnk} - H^i_{hr}(\mathcal{B}_m G^r_{jnk}) - (\mathcal{B}_r H^i_{kh})G^r_{jmn} + (\mathcal{B}_n H^r_{kh})G^i_{jmr}$

$-(\mathcal{B}_n H^i_{rk})G^r_{jmh} - (\mathcal{B}_n H^i_{hr})G^r_{jmk} = (\dot{\partial}_j a_{mn})H^i_{kh} + (\dot{\partial}_j b_{mn})(\delta^i_k y_h - \delta^i_h y_k)$.

Thus, we have

Theorem 2.8. In $G\beta R - BR - F_n$, the Berwald curvature tensor H^i_{jkh} is non-vanishing if and only if (2. 29) holds good.

Transvecting (2.29) by y^k , using (1.1c) , (1.1f), (1.1a) and (1.6), we get

(2.30) $(\mathcal{B}_m H^r_h)G^i_{jnr} + H^r_h(\mathcal{B}_m G^i_{jnr}) - (\mathcal{B}_m H^i_r)G^r_{jnh}$

$-H^i_r(\mathcal{B}_m G^r_{jnh}) - (\mathcal{B}_r H^i_h)G^r_{jmn} + (\mathcal{B}_n H^r_h)G^i_{jmr}$

$-(\mathcal{B}_n H^i_r)G^r_{jmh} = (\dot{\partial}_j a_{mn})H^i_h - (\dot{\partial}_j b_{mn})(\delta^i_h F^2 - y^i y_h)$.

In view of (2.17) and (2.30), we get

(2.31) $(\mathcal{B}_m H^r_h)G^i_{jnr} + H^r_h(\mathcal{B}_m G^i_{jnr}) - (\mathcal{B}_m H^i_r)G^r_{jnh}$

$-H^i_r(\mathcal{B}_m G^r_{jnh}) - (\mathcal{B}_r H^i_h)G^r_{jmn} + (\mathcal{B}_n H^r_h)G^i_{jmr}$

$-(\mathcal{B}_n H^i_r)G^r_{jmh} = (\dot{\partial}_j a_{\ell m})[H^i_h - H(\delta^i_h - l^i l_h)]$.

If

(2.32) $(\mathcal{B}_m H^r_h)G^i_{jnr} + H^r_h(\mathcal{B}_m G^i_{jnr}) - (\mathcal{B}_m H^i_r)G^r_{jnh} - H^i_r(\mathcal{B}_m G^r_{jnh})$

$-(\mathcal{B}_r H^i_h)G^r_{jmn} + (\mathcal{B}_n H^r_h)G^i_{jmr} - (\mathcal{B}_n H^i_r)G^r_{jmh} = 0$.

We have at least one of the following conditions

(2.33) a) $(\dot{\partial}_j a_{mn}) = 0$, b) $H^i_h = H(\delta^i_h - l^i l_h)$.

Putting $H = F^2 R$, $R \neq 0$, (2.33) may be written as

(2.34) $$H_h^i = F^2 R(\delta_h^i - l^i l_h).$$

Therefore the space is a Finsler space of scalar curvature

Thus , we have

Theorem 2.9. An $G\beta R - BR - F_n$, for $(n > 2)$ admitting $(\mathcal{B}_m H_h^r) G_{jnr}^i + H_h^r(\mathcal{B}_m G_{jnr}^i) - (\mathcal{B}_m H_r^i) G_{jnh}^r - H_r^i(\mathcal{B}_m G_{jnh}^r) - (\mathcal{B}_r H_h^i) G_{jmn}^r + (\mathcal{B}_n H_h^r) G_{jmr}^i - (\mathcal{B}_n H_r^i) G_{jmh}^r = 0$ is a Finsler space of scalar curvature provided $R \neq 0$ and the covariant tensor filed a_{mn} is not independent of the directional arguments.

3. Conclusion

1. The space whose defined by condition (2.3) is called generalized $\beta R -$ birecurrent Finsler space.
2. In $G\beta R - BR - F_n$, Berwald second covariant derivative of the $h(v)-$torsion tensor H_{kh}^i and the deviation tensor H_h^i is given by the equations (2.4) and (2.5) ,respectively.
3. In $G\beta R - BR - F_n$, the $R-$ Ricci tensor R_{jk} , the curvature vector H_k and the scalar curvature H are non-vanishing .
4. In $G\beta R - BR - F_n$, the $H-$Ricci tensor H_{jk} is given by equation (2.12) if and only if (2.13) holds good.
5. In $G\beta R - BR - F_n$ **,** the covariant tensor field b_{mn} is independent of the directional argument**s** if and only if the covariant tensor field a_{mn} is independent of the directional argument**s** provided $(\mathcal{B}_r H) G_{jmn}^r = 0$.
6. In $G\beta R - BR - F_n$, which the covariant tensor field a_{mn} is not independent of the directional argument at least one of the conditions (2.21a) and (2.21b) hold.
7. An $G\beta R - BR - F_n$ is Landsberg space if condition (2.21b) holds and provided $H(\mathcal{B}_m G_{knj}^r) y_r y^m = 0$
8. An $G\beta R - BR - F_n$ is necessarily Landsberg space provided $a_{jn} \neq \dot{\partial}_j a_n$ and $H(\mathcal{B}_m G_{knj}^r) y_r y^m = 0$
9. In $G\beta R - BR - F_n$, the Berwald curvature tensor H_{jkh}^i is non-vanishing if and only if (2. 29) holds good.
10. An $G\beta R - BR - F_n$, for $(n > 2)$ admitting $(\mathcal{B}_m H_h^r) G_{jnr}^i + H_h^r(\mathcal{B}_m G_{jnr}^i) - (\mathcal{B}_m H_r^i) G_{jnh}^r - H_r^i(\mathcal{B}_m G_{jnh}^r) - (\mathcal{B}_r H_h^i) G_{jmn}^r + (\mathcal{B}_n H_h^r) G_{jmr}^i - (\mathcal{B}_n H_r^i) G_{jmh}^r = 0$ is a Finsler space of scalar curvature provided $R \neq 0$ and the covariant tensor filed a_{mn} is not independent of the directional arguments.

4. Recommendations

The authors recommend the research should be continued in the Finsler spaces because it has many applications in Biology , relativity physics and other fields .

References

[1]. **A.G. Walker.** On Ruse's space of recurrent curvature, Proc. Land Math. Soc., 52 ,1950,pp. 36-64.

[2]. **B.B.Sinha** and **S.P.Singh ''** On recurrent spaces of second order in Finsler space." Yakohama Math. J.,vol. 18,pp. 27-32, 1970.

[3]. **F.Y.A .Qasem** and **A.A.M.Saleem''** On U- birecurrent Finsler space." Univ. Aden J. Nat. and Appl . Sc.Vol.14 No.3 ,December. 2010.

[4]. **H.S .Ruse.** Three dimensional spaces of recurrent curvature: Proc. Lond. Math. Soc., 50, 1949,pp. 438-446.

[5]. **P. N. Pandey.** A recurrent Finsler manifold admitting special recurrent transformation: Prog . Math.,13 (1-2) , 1979 ,pp. 85-98.

[6]. **P. N. Pandey .** A note on recurrence vector : proc. Nat. Acad. Sci. 51A,I , 1981,pp. 6-8.

[7]. **P. N. Pandey**." Some problems in Finsler spaces." D.Sc. Thesis, University of Allahabad, India ,1993.

[8]. **P.N. Pandey, S. Saxena,** and **A. Goswani ''**On a Generalized H-Recurrent space." Journal of International Academy of physical Science, Vol. 15, pp. 201-211, 2011.

[9]. **R .B. Misra.** On a recurrent Finsler spaces :Rev .Roumhaine Math. Pures Appl.18 ,1973,pp.701 - 712.

[10]. **R.S.D. Dubey** and **A.K. Srivastava.** On recurrent Finsler spaces: Bull. Soc.Math. Belgique ,1981,pp. 283-288.

[11]. **R**.**S**. **Mishra** and **H**.**D**. **Pande** "Recurrent Finsler space." J, India Math. Soc. N.S. vol.32, pp.17-22, 1968.

[12]. **R. Verma** "Some transformations in Finsler spaces." Ph.D. Thesis, University of Allahabad, ,India, 1991.

[13]. **S**. **Dikshit ''**Certain types of recurrences in Finsler spaces." Ph.D. Thesis, University of Allahabad, 1992 .

[14]. **Y .C. Worg .** Linear connections with zero torsion and recurrent curvature : Trans . Amer . Math. Soc ., 102 ,1962, pp.471 – 506.

[15]. **Y .C .Worg** and **K. Yano.** projectively flat spaces with recurrent curvature : Comment Math . Helv .,35 ,1961, pp. 223 – 232 .