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A Static Solution to Einstein's Field Equations for a Spherical Distribution of Electrically Counterpoised Dust

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Abstract

The work on Einstein-Maxwell equations for a sphere composed of a special kind of matter distribution called electrically counterpoised dust (ECD) with constant density has been discussed. The better understanding of the electrically counterpoised dust model in detail will helps scientists to understand the universe in better way. Along this line, the understanding of static solutions for the electrically counterpoised dust will be a crucial factor. To understand the static solutions for the electrically counterpoised dust, static solutions for the Einstein-Maxwell equations have been found. In particular, static solutions for a sphere composed of electrically counterpoised dust matter using standard boundary conditions tested. In addition to that the expressions for the redshift moving along a radial geodesic have been obtained.

Keywords: Einstein field equations; electrically counterpoised dust.

1. Introduction

In a distribution of electrically counterpoised dust, there are no resultant force between each particle since all the forces between particles in such a distribution have been balanced. Wickramasuriya [1] considered a spherical distribution of electrically counterpoised dust (ECD) and proved that Einstein's field equations can be written in a simple form for such a distribution. Using this form, a static solution to Einstein's field equations has been found with the application of standard boundary conditions.

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The red shift of a pulse of light which is emitted at a point inside the sphere as observed by an observer who is at a large distance in the exterior region is also calculated.

2. Material and the Method

Wickramasuriya [1] consider a spherically metric in the form

$$\boldsymbol{ds^2} = \boldsymbol{e^{2U}c^2dT^2} - \boldsymbol{e^{-2U}}(\boldsymbol{dR^2} + \boldsymbol{R^2 d\theta^2} + \boldsymbol{R^2}\sin^2\boldsymbol{\theta}\,\boldsymbol{d\phi^2}) \qquad (1)$$

and proved that Einstein's field equations for a spherically symmetric distribution of electrically counterpoised dust (ECD) can be written in the form

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{de^{-U}}{dR}\right) = -4\pi\rho e^{-3U} \qquad (2)$$

where $\boldsymbol{\rho}$ is the density of the sphere.

Using the transformations $\boldsymbol{y} = \boldsymbol{e^{-U}}$ and $\boldsymbol{R} = \boldsymbol{lx}$, equation (2) can be written in the form

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -4\pi\rho l^2 y^3 \qquad (3)$$

where $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{x})$ and $\boldsymbol{y}' = \frac{dy}{dx}$.

Now choose $\boldsymbol{l}$ such that $\boldsymbol{4\pi\rho l^2} = \boldsymbol{1}$. Then (3) reduces to

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^3 \qquad (4)$$

Equation (4) is in the form of Lane Emden equation, the solution of which is $\boldsymbol{y}$, which we write as $\boldsymbol{\theta}(\boldsymbol{x})$ since the solution of $\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^3$ is usually written as $\boldsymbol{\theta}(\boldsymbol{x})$, the so called Lane Emden function.

Lane Emden function has been expressed as a power series in even powers of $\boldsymbol{x}$ .The power series converges slowly, and it is more advantageous to obtain the solution in terms of a table of values according to Dharmawardane [2].

He has plotted the graph of $\boldsymbol{\theta}\left(\frac{R}{l}\right)$ and $\boldsymbol{\theta}\left(\frac{R}{l}\right) + \frac{R}{l}\boldsymbol{\theta}'\left(\frac{R}{l}\right)$ against $\frac{R}{l}$ which is reproduced in Figure 1, where $\boldsymbol{x} = \frac{R}{l}$

y

Now as it has been shown in Wimaladharma [3] that $e^{U(R)} = 1 \pm \Phi(R)$, where $\Phi(R)$ is the Newtonian potential divided by c^2, where c is the velocity of light.

Assuming that when $R = 0$, Φ and $\frac{\partial \Phi}{\partial R}$ to be equal to zero, due to physical conditions, it can be proved that $y(0) = 1$ and $y'(0) = 0$.

Therefore the initial of the equation (3) are the same as the initial conditions of Lane Emden equation.

As an illustration we have, for a few values of x, the values of the Emden function $\theta(x)$ given by Dharmawardane [2].

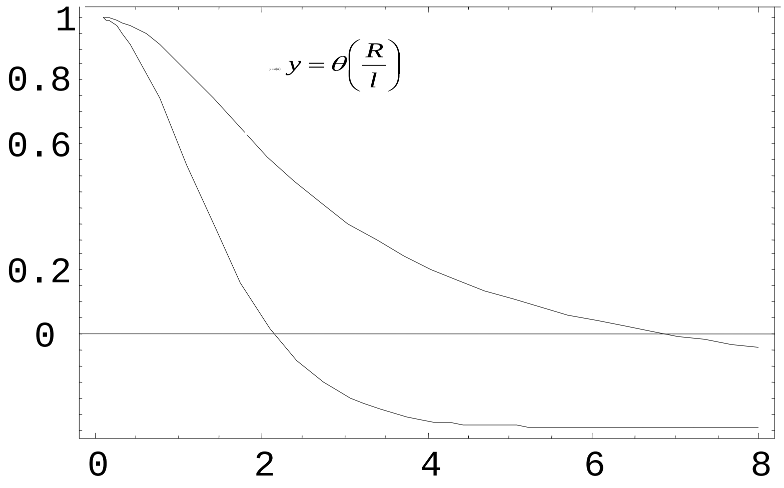

Figure 1: the graph of $\theta\left(\frac{R}{l}\right)$ and $\theta\left(\frac{R}{l}\right) + \frac{R}{l}\theta'\left(\frac{R}{l}\right)$ against $\frac{R}{l}$

In Figure 2 a graph of $\left(-\left(\frac{R}{l}\right)^2 \theta'\left(\frac{R}{l}\right)\right)$ is reproduced against $\frac{R}{l}$.

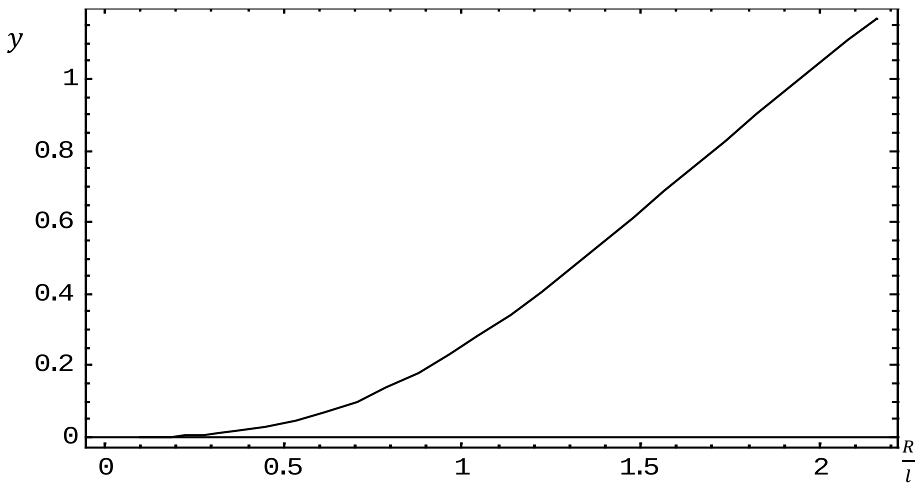

Figure 2: the graph of $\left(-\left(\frac{R}{l}\right)^2 \theta'\left(\frac{R}{l}\right)\right)$

Since $\theta(x) = y(x)$, the interior solution for the above distribution of electrically counterpoised dust (ECD) can be written as $e^{-U} = \theta\left(\frac{R}{l}\right)$, where $x = \frac{R}{l}$.

Therefore the metric for the interior region comprising electrically counterpoised dust (ECD) can be written in the form

Table 1: the values of the Emden function $\theta(x)$

x	$\theta(x)$
0.0	1.000000
1.0	0.855058
2.0	0.582851
3.0	0.359227
4.0	0.209282
5.0	0.11082
6.0	0.043738
7.0	-0.0043122

$$ds^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2} c^2 dT^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2 \left(dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2\right) \;, R \leq a \tag{5}$$

where a is the radius of the sphere.

Now for the exterior vacuum region, the matter density ρ is equal to zero since there is no matter present there.

Therefore, for the exterior vacuum region (2) becomes

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{de^{-U}}{dR}\right) = 0 \tag{6}$$

The solution of the differential equation (6) takes the form

$$e^{-U} = A_1 + \frac{A_2}{R} \tag{7}$$

where A_1 is a constant.

Therefore the corresponding exterior metric can be written as

$$ds^2 = e^{2U} c^2 dT^2 - e^{-2U}\left(dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2\right) \tag{8}$$

where $e^{-U} = A_1 + \frac{A_2}{R}$, $R > a$.

Now, from (5) and (8) the metric for the whole distribution of matter can be written as

$$ds^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2} c^2 dT^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2 (dR^2 + R^2 d\Omega^2) \ , \quad R \le a$$

$$ds^2 = \frac{1}{\left(A_1 + \frac{A_2}{R}\right)^2} c^2 dT^2 - \left(A_1 + \frac{A_2}{R}\right)^2 (dR^2 + R^2 d\Omega^2) \qquad a < R \tag{9}$$

where $d\Omega^2 = d\theta^2 + R^2 \sin^2\theta \, d\phi^2$.

Here we apply the standard boundary conditions, known as Lichernowicz's boundary conditions which says that metric coefficients and their derivatives are continuous at the boundary , $R = a$.

Then we obtain

$$\frac{1}{\theta\left(\frac{a}{l}\right)} = \frac{1}{\left(A_1 + \frac{A_2}{a}\right)} \tag{10}$$

$$-\frac{1}{\left(\theta\left(\frac{a}{l}\right)\right)^3}\left(\frac{1}{l}\theta'\left(\frac{a}{l}\right)\right) = -\frac{1}{\left(A_1 + \frac{A_2}{a}\right)^3}\left(-\frac{A_2}{a^2}\right) \tag{11}$$

Substituting (10) in (11), we obtain

$$A_2 = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right). \tag{12}$$

Using the fact that $A_2 = m$ according to Wickramasuriya [1], the value of the mass of the sphere of electrically counterpoised dust is found as

$$m = A_2 = -\frac{a^2}{l}\theta'\left(\frac{a}{l}\right) \tag{13}$$

Substitution of (13) in (10) gives

$$A_1 = \theta\left(\frac{a}{l}\right) - \frac{m}{a} . \tag{14}$$

Now the metric for the entire spherical distribution takes the form

$$ds^2 = \frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2} c^2 dT^2 - \left(\theta\left(\frac{R}{l}\right)\right)^2 (dR^2 + R^2 d\Omega^2) \ , \qquad 0 \le R \le a$$

$$ds^2 = \frac{1}{\left(\theta\left(\frac{a}{l}\right) - \frac{m}{a} + \frac{m}{R}\right)^2} c^2 dT^2 - \left(\theta\left(\frac{a}{l}\right) - \frac{m}{a} + \frac{m}{R}\right)^2 (dR^2 + R^2 d\Omega^2) \qquad a < R \tag{15}$$

According to the Figure 1, $\theta\left(\frac{a}{l}\right)-\frac{m}{a}=\theta\left(\frac{a}{l}\right)+\frac{a}{l}\theta'\left(\frac{a}{l}\right)=1$ only when $\frac{a}{l}=0$ which implies that

$a=0$ and $m=0$.

Thus the metric is Lorentzian at large distances only if a=0 and m=0, which makes the distribution devoid of matter. It is the flat space time without matter which is of no interest.

3. Results

3.1 The Red shift of a pulse of light

Here we consider the red shift of a pulse of light which is emitted at a point inside of the sphere as observed by an observer who is at a large distance away in the exterior region using expressions for null geodesics in General Relativity.

Consider a pulse of light with front emitted at $R=R_e$ at $T=T_e$ with frequency ν_e inside the sphere and an observer at $R=a$, just inside the surface of the sphere receiving it at $T=T_e{}'$ with frequency $\nu_e{}'$.

Now consider the radial null geodesics within the sphere.

$$0=\frac{1}{\left(\theta\left(\frac{R}{l}\right)\right)^2}c^2dT^2-\left(\theta\left(\frac{R}{l}\right)\right)^2dR^2$$

from which we obtain

$$\frac{dR}{dT}=\frac{c}{\left(\theta\left(\frac{R}{l}\right)\right)^2}\ . \qquad (16)$$

(The plus sign is taken since R increases with time T as the photons are going away from the centre of the sphere)

Now integration of (16) gives

$$\int_{T_e}^{T_e{}'}c\,dT=\int_{R_e}^{R_e{}'}\left(\theta\left(\frac{R}{l}\right)\right)^2dR \qquad (17)$$

Now consider the rear of the pulse emitted at $R=R_e$ at $T=T_e+\Delta T_e$ with frequency ν_e .

Let us assume that the rear of the pulse is observed at $R=a$, the boundary of the sphere at $T=T_e{}'+\Delta T_e{}'$ with frequency $\nu_e{}'$.

Integration of (17) for the rear of the pulse gives

$$\int_{T_e+\Delta T_e}^{T_e'+\Delta T_e'} c\, dT = \int_{R_e}^{R_e'} \left(\theta\left(\frac{R}{l}\right)\right)^2 dR \tag{18}$$

From (17) and (18) we obtain

$$\Delta T_e = \Delta T_e{}' \tag{19}$$

Now from the metric (15) the proper time intervals of the two observers corresponding to ΔT_e and $\Delta T_e{}'$ are given by

$$\Delta\tau_e = \left(\frac{1}{\theta\left(\frac{R_e}{l}\right)}\right)\Delta T_e \quad \text{and} \quad \Delta\tau_e{}' = \left(\frac{1}{\theta\left(\frac{a}{l}\right)}\right)\Delta T_e{}' \tag{20}$$

where $\Delta\tau_e$ and $\Delta\tau_e{}'$ are the proper time intervals of the two observers at $R = R_e$ and $R = a$, just inside the surface of the sphere, respectively, emitting and receiving the pulse.

Since the number of cycles of the pulse remain the same at emission and observation, we have

$$\nu_e \Delta\tau_e = \nu_e' \, \Delta\tau_e{}' \tag{21}$$

where ν_e and ν_e' are the emitted and observed frequencies respectively.

Then the equations (20) and (21) give

$$\frac{\nu_e'}{\nu_e} = \frac{\Delta\tau_e}{\Delta\tau_{e'}} = \frac{\theta\left(\frac{a}{l}\right)}{\theta\left(\frac{R_e}{l}\right)}\left(\frac{\Delta T_e}{\Delta T_{e'}}\right) \tag{22}$$

The equation (19) leads the equation (22) to

$$\frac{\nu_e'}{\nu_e} = \frac{\theta\left(\frac{a}{l}\right)}{\theta\left(\frac{R_e}{l}\right)} \tag{23}$$

Now consider the moment at which the pulse of light passes through the boundary.

Earlier in this section, we assumed that the observer at $R = a$, at the boundary of the sphere but inside it observed the pulse at $T = T_e{}'$ with frequency $\nu_e{}'$. Now our assumption is that an observer at $R = a$ but outside of the sphere observed the pulse at $T = T_0{}'$ with frequency $\nu_0{}'$.

Then using the metrics in (15), the proper times of the two observers corresponding to $\Delta\tau_e{}'$ and $\Delta\tau_0{}'$ can be written as

$$\Delta\tau_e{}' = \left(\frac{1}{\theta\left(\frac{a}{l}\right)}\right)\Delta T_e{}' \quad \text{and} \quad \Delta\tau_0' = \left(\frac{1}{\theta\left(\frac{a}{l}\right) - \frac{m}{a} + \frac{m}{a}}\right)\Delta T_0{}' = \left(\frac{1}{\theta\left(\frac{a}{l}\right)}\right)\Delta T_0{}' \tag{24}$$

where $\Delta\tau_0{}'$ and $\Delta\tau_e{}'$ are the proper time intervals of the two observers at $R = a$, inside the sphere and outside the sphere respectively observing the pulse.

Since the number of cycles in the pulse remain the same as it crosses the boundary, from the equation (24) we obtain

$$\frac{\upsilon_{e'}}{\upsilon_{0'}} = \frac{\Delta\tau_{0'}}{\Delta\tau_{e'}} = \frac{\Delta T_{0'}}{\Delta T_{e'}} \qquad (25)$$

According to the standard boundary conditions, the coordinate time intervals do not vary across the boundary.

Then the equation (25) simplifies to

$$\frac{\upsilon_{e'}}{\upsilon_{0'}} = 1 \qquad (26)$$

Now assume that an observer at a large distance $R = R_0$, outside of the sphere observed the front of the pulse which is passed through the boundary at $T = T_0{}'$ outside the sphere at $T = T_0$, and the rear of the pulse which is passed through the boundary at $T = T_0' + \Delta T_0{}'$ outside the sphere, at $T = T_0 + \Delta T_0$ respectively with the frequency υ_0.

Now we have to consider the exterior metric in (15) which is

$$ds^2 = \frac{1}{\left(\theta\left(\frac{a}{l}\right) - \frac{m}{a} + \frac{m}{R}\right)^2} c^2 dT^2 - \left(\theta\left(\frac{a}{l}\right) - \frac{m}{a} + \frac{m}{R}\right)^2 (dR^2 + R^2 d\Omega^2).$$

Considering the null geodesics of the metric we obtain

$$c\, dT = \left(\theta\left(\frac{a}{l}\right) - \frac{m}{a} + \frac{m}{R}\right)^2 dR \qquad (27)$$

(Plus sign is taken since the null ray is away from the centre of the sphere.)

Integration of (27) gives

$$\int_{T_0{}'}^{T_0} c\, dT = \int_{R=a}^{R_0} \left(\theta\left(\frac{a}{l}\right) - \frac{m}{a} + \frac{m}{R}\right)^2 dR \qquad (28)$$

If the rear of the pulse is observed at $T_0 + \Delta T_0$ at $R = R_0$, then integration of (27) gives

$$\int_{T_0' + \Delta T_0{}'}^{T_0 + \Delta T_0} c\, dT = \int_{R=a}^{R_0} \left(\theta\left(\frac{a}{l}\right) - \frac{m}{a} + \frac{m}{R}\right)^2 dR \qquad (29)$$

From the equations (28) and (29) we obtain

$$\Delta T_0' = \Delta T_0 \tag{30}$$

Then using the metrics in (15), the proper times of the two observers corresponding to $\Delta T_0'$ and ΔT_0 can be written as

$$\Delta\tau_0' = \left(\frac{1}{\theta\left(\frac{a}{l}\right)}\right)\Delta T_0' \quad \text{and} \quad \Delta\tau_0 = \left(\frac{1}{\theta\left(\frac{a}{l}\right)-\frac{m}{a}+\frac{m}{R_0}}\right)\Delta T_0 \tag{31}$$

where $\Delta\tau_0'$ and $\Delta\tau_0$ respectively are the proper time intervals of the two observers at $R = a$,outside the sphere, and at $R = R_0$, a large distance outside of the sphere observing the pulse .

Since the number of cycles of the pulse remain the same throughout its way, we obtain

$$\frac{\upsilon_0}{\upsilon_0{}'} = \frac{\Delta\tau_0'}{\Delta\tau_0} \tag{32}$$

Substitution (31) in (32) gives

$$\frac{\upsilon_0}{\upsilon_0{}'} = \frac{\left(\theta\left(\frac{a}{l}\right)-\frac{m}{a}+\frac{m}{R_0}\right)}{\left(\theta\left(\frac{a}{l}\right)\right)}\left(\frac{\Delta T_0'}{\Delta T_0}\right) \tag{33}$$

Substituting $\Delta T_0' = \Delta T_0$ in (33), we obtain

$$\frac{\upsilon_0}{\upsilon_0{}'} = \frac{\left(\theta\left(\frac{a}{l}\right)-\frac{m}{a}+\frac{m}{R_0}\right)}{\left(\theta\left(\frac{a}{l}\right)\right)} \tag{34}$$

Using the fact $\frac{\nu_0}{\nu_e} = \frac{\upsilon_0}{\upsilon_0{}'}\frac{\upsilon_0{}'}{\nu_e{}'}\frac{\nu_e{}'}{\nu_e}$ and the equations (23), (26) and (34), we obtain

$$\frac{\nu_0}{\nu_e} = \frac{\left(\theta\left(\frac{a}{l}\right)-\frac{m}{a}+\frac{m}{R_0}\right)}{\left(\theta\left(\frac{a}{l}\right)\right)}\frac{\left(\theta\left(\frac{a}{l}\right)\right)}{\left(\theta\left(\frac{R_e}{l}\right)\right)} \tag{35}$$

Since the red shift z corresponds to the change of the wave length, we have

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{\theta\left(\frac{R_e}{l}\right)}{\left(\theta\left(\frac{a}{l}\right)-\frac{m}{a}+\frac{m}{R_0}\right)} \tag{36}$$

It is clear that the red shift as observed by an observer at infinity is given by

$$1 + z = \frac{\lambda_0}{\lambda_e} = \frac{\theta\left(\frac{R_e}{l}\right)}{\left(\theta\left(\frac{a}{l}\right)-\frac{m}{a}\right)} \tag{37}$$

Since $R_e < a$ and $\theta\left(\frac{R}{l}\right)$ is a decreasing function, it can be shown that the pulse of light is red shifted.

4. Conclusion

We obtained the metrics for an electrically counterpoised dust sphere using what are known as standard boundary conditions. In this formalism we use the same coordinates in the two regions inside the spherical distribution of matter as well as outside.

Also it is shown that a pulse of light which is emitted from the inside of the sphere as observed by an observer at infinity is red shifted. So the solution obtained for the Einstein's field equations is physically acceptable.

However, the metric we obtained for the external region is not Lorentzian at infinity (large distances) in general and it becomes Lorentzian only when a, the radial coordinate and m, the mass of the ECD distribution becomes zero. That is when there is no matter distribution and which makes the space time flat and Lorentzian throughout.

In our next article, we will overcome this difficulty by using different coordinates for the two regions, inside of the sphere and outside of the sphere. Then we will require more equations to solve Einstein's field equations. Therefore we will apply a new set of boundary conditions other than standard boundary conditions to find exact solutions to Einstein's field equations. Also we are going to compare the results obtained in this article with the results which will be obtained using new set of boundary conditions.

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