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A High Accurate Approximation for a Galactic Newtonian Nonlinear Model Validated by Employing Observational Data

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Abstract

This article proposes Perturbation Method (PM) to solve nonlinear problems. As case study PM is employed to provide a detailed study of a nonlinear galactic model. Our approach is rather elementary and seeks to explain as much detail as possible the material of this work.

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In particular, our solution gives rise qualitatively, to the known flat rotation curves. In fact, we compare the numerical solution and the obtained approximation by employing observational data proving the validity and high accuracy of the model under study.

Keywords: Perturbation method; nonlinear galactic model; flat rotation curves; approximated solutions.

1. Introduction

The unusual behavior of the rotation curves in spiral galaxies is a scientific discovery of paramount importance. It has become one of the strongest arguments supporting the existence of dark matter and has even led to proposing modifications of Newtonian gravity and general relativity theory. Nevertheless, instead of considering the above approaches here, this paper considers the possibility to explain them without leaving the scope of Newtonian physics.

Essentially a spiral galaxy consists of:

1) Disk. In this region, stars and other components such as gases essentially move in circular orbits around the center of the galaxy. This part is characterized for the energy of motion mainly due to rotation, in fact the random movements represent only 10% of that value [2].

2) Stellar halo. Is an extended, roughly spherical component of a galaxy which extends beyond the main, visible component [2]. This region is characterized by disordered rotational movements and large random motions.

3) Bulge. Represents the central part of the galaxy. The most significant fact of this region is that the stars that compose it rigidly rotate with constant angular velocity.

From the above, we highlight an important fact; the differential character of the rotation. It is not difficult to deduce that the rotation curve (i.e, the curve representing the speed of rotation as a function of distance from the center of mass of the galaxy) should indicate that up to a certain distance from the center, the orbital velocity, increases proportionally with this, while away from the disk (at a distance at which it is not observed mass) should occur a Keplerian rotation, that is, a planet-like rotation [2,7]. Instead, it is observed that the rotational speed remains essentially constant, i.e. the angular velocity must decrease with radius, as if the mass of the galaxy following increasing with distance from the galactic center. This matter, which is supposed does not emit light and is contained in the halo, is called dark matter.

Although the dark matter scenario in astrophysics has been widely employed, [1] proposed a nonlinear model of Newtonian gravity, without resorting to dark matter and other theoretical modifications.

The aim of our approach is reframe the mentioned work, with a rather elementary view, and seeks to explain as much detail as possible the material of this article. Unlike the mentioned paper [1], from the beginning we restrict the application of the model, to the disk of the galaxy, emphasizing the perturbative character of the equation to solve and justifying the accuracy of the obtained results.

This paper is organized as follows. In Section 2, we introduce the basic idea of PM method. For Section 3, we provide an application of PM method, by solving the differential equation which describes a galactic nonlinear model. Section 4 discusses the main results obtained, while, a brief conclusion is given in Section 5. Finally, appendix A provides observational data, to perform numerical comparisons.

2. Basic idea of perturbation method

The perturbation method (PM) is a well-established method; it is among the pioneer techniques to approach various kinds of nonlinear problems. This procedure was originated by S. D. Poisson and extended by J. H. Poincare. Although the method appeared in the early 19th century, the application of a perturbation procedure to solve nonlinear differential equations was performed later on that century. The most significant efforts were focused on celestial mechanics, fluid mechanics, and aerodynamics [3-6].

Let the differential equation of one dimensional nonlinear system be in the form

$$L(x)+\varepsilon N(x)=0, \qquad (1)$$

where we assume that x is a function of one variable $x=x(t)$, $L(x)$ is a linear operator which, in general, contains derivatives in terms of t, $N(x)$ is a nonlinear operator, and ε is a small parameter.

Considering the nonlinear term in (1) to be a small perturbation and assuming that the solution for (1) can be written as a power series in the small parameter ε,

$$x(t)=x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+... \qquad (2)$$

Substituting (2) into (1) and equating terms having identical powers of ε, we obtain a number of differential equations that can be integrated, recursively, to find the values for the functions: $x_0(t)$, $x_1(t)$, $x_2(t)$ …

3. Approximate solution of a nonlinear differential equation which describes a galactic nonlinear model.

We start from the fundamental consideration that the speed of rotation of matter in a galaxy at a distance r from the galactic center, must be related to the potential acting on it. Thus, following Newtonian theory, the potential is given by Poisson equation

$$\nabla^2 V(\vec{r},t)=4\pi G\rho(\vec{r},t), \qquad (3)$$

where $G=6.67384\times10^{-11}Nm^2/kg^2$, is the gravitational constant.

Therefore, the rotational speed requires through (3) a detailed knowledge of the mass density $\rho(\vec{r},t)$.

To have an adequate representation of the galactic density, we will take into account that the experimental data show that the mass distribution is Gaussian in the Observed velocities (that is $\approx e^{-\alpha v^2}, \alpha =$ constant) [1]. This allows conceive a spiral galaxy as a fluid with a given mass distribution in the phase space, say $\chi(\vec{r},t)$. In accordance with [1,8], the integral of this last over the velocities is related to density $\rho(\vec{r},t)$ through the following integral

$$\rho(\vec{r},t) = m\int \chi(\vec{r},\vec{v},t)d\vec{v}\ , \qquad (4)$$

where m is the mass of a typical star.

Following [8], mass distribution is also Gaussian in the kinetic energy (since kinetic energy and v^2 differ just in a constant factor), and therefore is reasonable to propose that the mass distribution χ to be a Boltzmann distribution in the total energy E [1]. In what follows, we will focus to the study of the disc of galaxy assuming a static regime. Also as we will see later, our results will be consistent with the behavior of the rotation curves of elliptical galaxies.

From the above,

$$\chi(E) = \chi_0 e^{-\beta E}, \qquad (5)$$

for some parameters χ_0 and β (in the context of statistical mechanics β would be identified with the so called, temperature parameter [8]).

In order to exploit the axial symmetry of the system, we will employ cylindrical coordinates, so that it is possible to rewrite (5) as

$$\chi(E) = \chi_0 \exp[-\beta\left(\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + V(r,z)\right), \qquad (6)$$

where it was considered, the energy per unit mass, to account for the presence of potential rather than the potential energy. Similarly it is assumed that the total energy is mechanical, due to conservative character of gravitational force field [3,7].

Thus, substituting (6) into (4), we obtain

$$\rho(r,z) = m\int rd\dot{r}d\dot{\phi}d\dot{z}\chi_0 \exp\left[-\beta\left(\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) + V(r,z)\right)\right], \qquad (7)$$

To evaluate (7) over the velocities, we employ Gaussian integrals, so that the sought dependence of the mass density over the gravitational potential is given by

$$\rho(r,z) = m\chi_0 \left(\frac{2\pi}{\beta}\right)^{3/2} e^{-\beta V(r,z)}, \qquad (8)$$

Substituting (8) into (3), we get

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 4\pi G \rho_0 e^{-\beta V(r,z)}, \qquad (9)$$

where, we expressed the laplacian in cylindrical coordinates and defining

$$\rho_0 = m\chi_0 \left(\frac{2\pi}{\beta}\right)^{3/2}. \qquad (10)$$

Given the complexity of nonlinear partial differential equation (9), Reference [1] studied two over simplified models. The first case assumed that V varies only with the height to the plane of the disc, and despite the nonlinear character of the resultant equation, got at exact solution.

On the other hand, the second case study, assumed a variation only in the radial direction, i.e. $V(r)$.

The methodology used, expressed the resultant equation in terms of a Volterra second order integral equation, which was solved by means of Piccard´s method, but [1] did not discussed about the perturbative character of the differential equation for $V(r)$, and did not provide a manner to measure the accuracy of the proposed results.

Therefore, unlike the above methodology, this work will employ an understandable elementary procedure, based on classical perturbation method (PM), in order to find an analytical approximate solution for the above mentioned model which besides, takes into account the perturbative character of the equation to solve. We will see that our results coincide with those obtained in [1] but besides, we will provide the residual error in order to make sure of the accuracy of the proposed methodology.

Thus, (9) adopts the following form for the case where the potential varies only in the radial direction.

$$\frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 4\pi G \rho_0 e^{-\beta V(r)}. \qquad (11)$$

It's well-known that the typical density of a galaxy is low, of the order $1\times10^{-22}\, k/m^3$, and therefore

$$\varepsilon = 4\pi G\rho_0 \cong 8.38\times 10^{-32} \qquad (12)$$

Thus $\varepsilon << 1$, and it works as perturbation parameter; for the same, PM scheme explained in section 2 is adequate for this case.

From the above, (11) gives rise to

$$V''(r) + \frac{1}{r}V'(r) - \varepsilon e^{-\beta V(r)} = 0, \qquad V(r_0) = V_0, V'(r_0) = b. \qquad (13)$$

where prime denotes from here on, differentiation respect to r , r_0 is the radius of the galactic core and V_0 , b denotes the initial conditions of the problem (See Appendix A for more details).

We identify (see (1)).

$$L(r) = V'' + \frac{1}{r}V'; N(r) = -e^{-\beta V(r)}. \qquad (14)$$

Identifying ε with the PM parameter, we assume a solution for (13) in the form

$$V(r) = \nu_0(r) + \varepsilon\nu_1(r) + \varepsilon^2\nu_2(r) + \varepsilon^3\nu_3(r) + \varepsilon^4\nu_4(r) + ..., \qquad \text{(see (2)).} \qquad (15)$$

On comparing the coefficients of like powers of ε it can be solved for $\nu_0(r)$, $\nu_1(r)$, $\nu_2(r)$, $\nu_3(r)$,..and so on. Later it will be seen that, a very accurate handy result is obtained, by keeping up to first order approximation.

$$\varepsilon^0)\ \nu_0'' + \frac{1}{r}\nu_0' = 0,\ \nu_0(r_0) = V_0, \qquad \nu_0'(r_0) = b, \qquad (16)$$

$$\varepsilon^1)\ \nu_1'' + \frac{1}{r}\nu_1' - e^{-\beta\nu_0} = 0,\ \nu_1(r_0) = 0, \quad \nu_1'(r_0) = 0. \qquad (17)$$

To find the solution of (16) that satisfies the initial conditions, we note that it, is indeed a Cauchy-Euler equation [9]

$$r^2\nu_0'' + r\nu_0' = 0, \qquad (18)$$

therefore

$$\nu_0(r) = V_0 + r_0 b \ln\left(\frac{r}{r_0}\right), \qquad (19)$$

Substituting (19) into (17) is obtained

$$v_1'' + \frac{1}{r} v_1' = e^{-\beta V_0} \left(\frac{r}{r_0} \right)^{-r_0 b \beta}, \; v_1(r_0) = 0, \quad v_1'(r_0) = 0. \tag{20}$$

To solve (20), we employ the variation of parameters method [9] which requires evaluating the following integrals

$$u_1 = -\int \frac{f(r) \ln r dr}{W}, u_2 = \int \frac{f(r) dr}{W}, \tag{21}$$

where $y_1 = 1$ and $y_2 = \ln r$ are the solutions of the homogeneous differential equation

$$v_1'' + \frac{1}{r} v_1' = 0, \tag{22}$$

W is the Wronskian of these two functions, which is given by

$$W(y_1, y_2) = 1/r, \tag{23}$$

and $f(r)$ is the right hand side of (20).

By substituting $f(r)$ and (23) into (21) leads to

$$u_1 = \frac{r_0^{r_0 c \beta}}{r^{r_0 c \beta - 2}} e^{-\beta V_0} \left[\frac{-\ln r}{2 - \beta r_0 b} + \frac{1}{(2 - \beta r_0 b)^2} \right], \tag{24}$$

$$u_2 = \frac{e^{-\beta V_0}}{2 - \beta r_0 b} \frac{r_0^{r_0 c \beta}}{r^{r_0 c \beta - 2}}, \tag{25}$$

where the above integration process, requires that $-r_0 b \beta \neq -1$.

Therefore, the solution of (20) is written according to method of variation of parameters as

$$v_1(r) = A + B \ln r + \frac{r_0^{r_0 b \beta} e^{-\beta V_0} r^{2 - \beta r_0 b}}{(2 - \beta r_0 b)^2}, \tag{26}$$

To determine the values of A and B, we apply the boundary conditions $v_1(r_0) = 0$ and $v_1'(r_0) = 0$ to (26) to get

$$A = {r_0}^2 e^{-\beta V_0} \left[\frac{\ln r_0}{2-\beta r_0 b} - \frac{1}{\left(2-\beta r_0 b\right)^2} \right], \tag{27}$$

$$B = \frac{-e^{-\beta V_0} {r_0}^2}{2-\beta r_0 b}, \tag{28}$$

Thus, by substituting (19) and (26) into (15) we obtain a first order approximation for the solution of (13), as it is shown

$$V(r) = V_0 + \left[b r_0 - \frac{\varepsilon {r_0}^2 e^{-\beta V_0}}{2-\beta r_0 b} \right] \ln\left(\frac{r}{r_0}\right) + \frac{\varepsilon {r_0}^{r_0 b \beta} e^{-\beta V_0}}{\left(2-\beta r_0 b\right)^2} \left(r^{2-\beta r_0 b} - {r_0}^{2-\beta r_0 b} \right). \tag{29}$$

It's important to note that (29) coincides with the obtained solution from [1], by resorting to a more straightforward procedure.

In order to show that (29) gives rise qualitatively to the observed galaxy rotation curves; we note that the speed of rotation of a body of mass m in a galaxy at a distance r from the galactic center, must be related to the gravitational potential acting on it. From Newtonian theory, the gravitational force F_g, provides the centripetal acceleration $mv^2 / r = F_g$, or

$$\frac{v^2}{r} = g, \tag{30}$$

where g, is the magnitude of the intensity of gravitational field [2,3,7]. Likewise g, is related to the gravitational potential, according to $g = dV / dr$, so that

$$v = \sqrt{r \frac{dV(r)}{dr}}. \tag{31}$$

After differentiating (29) and substituting this result into (31), we obtain

$$v = \sqrt{b r_0 + \frac{\varepsilon {r_0}^2 e^{-\beta V_0}}{2-\beta r_0 b} \left[\left(\frac{r}{r_0}\right)^{2-\beta r_0 b} - 1 \right]}. \tag{32}$$

We note that if $\beta r_0 b > 2$, the model predicts qualitatively, the possibility of flat rotation curves, since in this case $\left(r/r_0\right)^{2-\beta r_0 b} \to 0,$ if $r \to \infty$ and (32) becomes in

$$v \to \sqrt{br_0 + \frac{\varepsilon r_0{}^2 e^{-\beta V_0}}{\beta r_0 b - 2}}\,. \qquad (33)$$

We see that (33) is consistent with the assumption $-r_0 b\beta \neq -1,$ since we have assumed that $\beta r_0 b > 2$ (see below equation (25)). Besides such as we mentioned, we observe that b can takes a wide range of values, without modifying both, the overall qualitative nature of the solution of (13) ((29)), neither the limit (33), assuming that $\beta r_0 b > 2$.

4. Discussion

This work emphasized two important aspects. From the mathematical point of view the proposed method PM, described with accuracy the model described by equation (14). Figure 1 shows the comparison between approximation (29) for (13) with the numerical solution (for comparison purposes, we considered that the "exact" solution is computed using a Fehlberg fourth-fifth order Runge–Kutta method with degree four interpolant (RKF45) [10, 11] as a build-in routine from Maple 17. Moreover, the routine was configured using an absolute error (A.E.) tolerance of $1x10^{-12}$), where we employed typical observational values for the parameters of (29) (we remit the above calculations to the appendix A). Although the figures are in good agreement, showing the highly accuracy of (29), it was verified by calculating the residual error (R.E) of (29). R.E is obtained substituting (29) into (13). Figure 2 shows that the maximum value of R.E is $<< 1\times 10^{-50}$ which confirms quantitatively the accuracy of the proposed solution. Although Figure 1 shows (29) in $[1.42\times 10^{20}, 10^{23}]$, from the Figure 2 is clear that the high accuracy of our approximate solution continues beyond. The methodology used by [1] consisted in expressing the resulting equation (11) in terms of a Volterra second order integral equation, which was solved by means of Picard's method, but the aforementioned paper lack of discussion about the perturbative character of the differential equation for $V(r)$ and it did not provide a manner to measure the accuracy of the proposed results, since the observational data were not taken into account. Unlike the above methodology, this work employed an elementary procedure based on classical perturbation method in order to find an analytical approximate solution for (13). A relevant fact of the proposed PM solution was highlight the perturbative character of the equation to solve. Although the results of [1] coincide with ours, it did not give justification to truncate the solution by keeping just the first Picard iteration. In fact we proved the feasibility of the model given by (13) and (29), by showing that it works employing reasonable observational data. From the astrophysicist point of view, this work is important because it shows that the qualitative behaviour of the elliptical galaxies rotation curves, can be explained in terms of a nonlinear model of the gravitational field, which preserves the validity of Newtonian physics and does not require the existence of dark matter. Unlike [1], this paper used basic physical concepts, to express their ideas like mechanical energy, centripetal force, kinetic energy and conservative field, so that it is expected be more understandable.

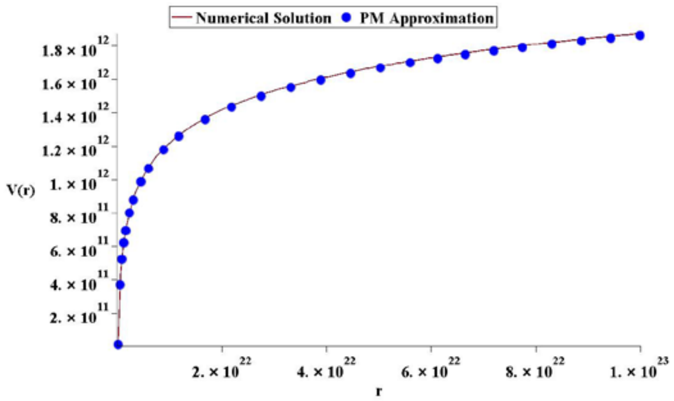

Figure 1: Comparison numerical solution of the nonlinear problem given by (13) and PM approximation (29) for the values of the parameters given by (1-A)-(5-A).

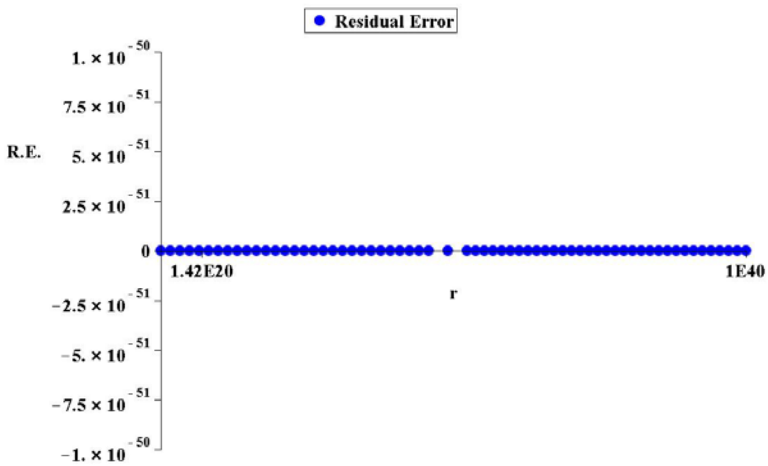

Figure 2: Residual error (R.E.) of (29) for the values of the parameters given by (1-A)-(5-A).

5. Conclusions

This paper presented a nonlinear model for the gravitation field that correctly describes the behaviour of the galaxies rotation curves and fits with some typical observational data, without departing from the Newtonian scope known. A highlight of this work is that although this is based on elementary mathematics and physics, very accurate results were obtained.

Appendix A

Detailed Calculations for Numerical Comparison.

Such as it was already deduced, the value of the perturbation parameter is (see (12))

$$\varepsilon \cong 8.38 \times 10^{-32} \qquad (1\text{-}A)$$

In order to get an estimation for $V'(r_0)$ we rewrite (31) in the following form

$$\frac{v^2}{r} = V'(r),$$

Taking as a typical values for $r_0 = 11000$ light years $\left(1.05\times10^{20}\,mts\right)$ and $v = 4.6\times10^5\,m/s$ (velocity at the bulge's border), we obtain

$$b = V'(r_0) = 2\times10^{-9}\ j/kg\cdot m \qquad \textit{(2-A)}$$

From the above

$$br_0 = 2.1\times10^{11}\ j/kg \qquad \textit{(3-A)}$$

On the other hand, with the purpose of providing an estimation of V_0, we assume a body of mass m, to a distance r_0 from the center of the galaxy.

From Newton second law, we get

$$\frac{mv^2}{r_0} = \frac{GMm}{r_0^{\,2}}, \text{ or}$$

$$v^2 = \frac{GM}{r_0},$$

that is

$$V_0 = V(r_0) = 2.1\times10^{11}\ j/kg. \qquad \textit{(4-A)}$$

Finally, as a illustrative example, we consider

$$\beta = 1 \qquad \textit{(5-A)}$$

Thus, by substituting (1-A) – (5-A) into (29) we get a first order approximation for the solution of (13), valid for the above values.

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References

[1] C. A. M. de Melo and S. T. "Resende. Galactic Nonlinear Dynamical Model." International Journal of Modern Physics A. World Scientific, 2009,

[2] Almudena Zurita Muñoz. "Curso Cosmología. Curvas de rotación y materia oscura en galaxias espirales." Internet:

http://www.ugr.es/~azurita/docencia/material_docente/cosmologia/cr_dm_clase_06.pdf , 2007 [Sep. 2015]

[3] T.L. Chow. Classical Mechanics. USA: John Wiley and Sons Inc., 1995.

[4] M.H. Holmes. Introduction to Perturbation Methods. New York: Springer-Verlag, 1995.

[5] U. Filobello-Niño, H. Vazquez-Leal, Y. Khan, A. Yildirim, V.M. Jimenez-Fernandez, A. L Herrera-May, R. Castañeda-Sheissa, and J. Cervantes-Perez. "Using Perturbation Method and Laplace-Padé Approximation to solve nonlinear problems." Miskolc Mathematical Notes, vol. 14(1), pp. 89-101, 2013. ISSN: 1787-2405,

[6] U. Filobello-Niño, H. Vazquez-Leal, K. Boubaker, Y. Khan, A. Perez-Sesma, A. Sarmiento Reyes, V.M. Jimenez-Fernandez, A Diaz-Sanchez, A. Herrera-May, J. Sanchez-Orea and K. Pereyra-Castro. "Perturbation Method as a Powerful Tool to Solve Highly Nonlinear Problems: The Case of Gelfand's Equation." Asian Journal of Mathematics and Statistics, vol. 2013, pp. 7, 2013. DOI: 10.3923 /ajms.2013, ISSN 1994-5418, 2013.

[7] R. Resnick and D. Halliday. Física Vol. 1, USA: John Wiley and Sons, 1977.

[8] F. Mandl. Física Estadística. México: Editorial Limusa, S.A, 1979.

[9] Dennis Zill. A First Course in Differential Equations with Modeling Applications, USA: Brooks /Cole Cengage Learning, 10th Edition, 2012

[10] W. Enright, K. Jackson, S.A. Norsett, P. Thomsen. "Interpolants for Runge–Kutta formulas." ACM Transactions on Mathematical Software (TOMS), vol. 12(3) pp. 193-218, 1986.

[11] E. Fehlberg. "Klassische Runge–Kutta–Formeln vierter und niedrigerer ordnung mit schrittweitenkontrolle und ihre anwendung auf Waermeleitungsprobleme." Computing, vol. 6, pp. 61-71, 1970