

# An Aperiodic Stable Chaos with Lyapunov Exponents in Time Series

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## Abstract

A new formula is developed to reproduce the shape of energy profiles for aperiodic stable attractors with Lyapunov exponents,  $\pm n_f$  by using the Fractional Fourier Transform (FRFT), i.e.

$$\psi_{\pm n_f}(\pm \omega_o t_{att}) = \left( \frac{\omega_o t}{\sqrt{\pi}} \right)^{\frac{1}{2}} 2^{\pm \frac{n_f}{2}}$$

where  $\frac{\pi}{2} < \omega_o t \leq 2\pi$ ,  $0 \leq \omega_o t \leq \frac{\pi}{2}$ ,  $\omega_o$  is the initial angular frequency of the of the attractor and  $t_{att}$ , the time of flight of the attractor. With  $\omega_o t = \frac{\pi}{2}$ , the energy profile for periodic unstable attractors at different values of Lyapunov exponents  $\pm n_f$  is obtained, for  $\frac{\pi}{2} < \omega_o t \leq 2\pi$  and  $0 \leq \omega_o t \leq \frac{\pi}{2}$  aperiodic stable attraction at different values of Lyapunov exponents  $\pm n_f$  are observed. The critical analysis about chaos is presented with emphasis to time series modeling and simulation.

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**Keywords:** Fractional Fourier Transform; Lyapunov Exponents; Aperiodic stable Chaos; Periodic instable attractors.

## 1. Introduction

Deterministic nonlinear model techniques of complex data series, i.e., uncorrelated series with a flat Fourier Spectrum have received a great deal of attention because it can be used to forecast the evaluation of a chaotic system [1]. The dynamic variables are usually unknown in chaotic systems. The cross-correlation function between observed and predicted values from nonlinear model techniques such as ARIMA (Auto Regressive Integrated Moving Average) provides estimated values of Lyapunov exponents of dynamic variables even for sparse time series (containing of the order of  $10^3$  data points). When the fit is achieved by using nonlinear modeling, we can say it is better than probabilistic models provided a deterministic mechanism governs the process under study [2,3]. Most of the non-linear modeling techniques follow Non-Bayesian statistics and are grouped in to major classes: global and local. The local non-Bayesian statistics has more advantages as compared to global non-Bayesian statistics. In the Bayesian approach [4] one assumes that the prior uncertainty about unknown parameters which have to be inferred from random data or from a stochastic process, can also be encoded in a probability distribution, the so called prior. The problem of distinguishing chaos from correlated noise or combinations of deterministic and randomness is a more difficult task [5]. Jafri [6] exploited concepts of Bayesian and non-Bayesian statistics to prove mathematically that the chaotic time series is deterministic. She assumed that the optimum metric (used as tool to distinguish chaos from correlated noise) arises from a metric tensor whose components are

$$\delta_{ij} = \partial_{ij} e^{2ic} \quad (1)$$

where  $\delta_{ij}$  is the kronecker delta function and  $i$  and  $j$  run from  $1$  to  $d$  (embedding dimension on the prediction interval) for input-output data pairs. If the parameter  $c$  in equation (1) is varied to minimize the root mean square error of the forecast, then there is a single global minimum corresponding to a value of  $c$  closed to the most negative Lyapunov exponent of the dynamics. Chaos is having an impact on diverse discipline of knowledge including physics, biology, chemistry, economics and medicine [7,12]. The chaos may behave almost linearly in some part of phase space and highly non-linearly in other parts. Lorenz and Rossele [13,14], identified non-stable chaos approximately in to different (chaotic and non-chaotic) nonlinear dynamical system with their attractors. But, the recognition of attractors in a chaotic system is indeed difficult. Jafri [15,16,21], identified an aperiodic stable chaos with an attractor in Mackey- Glass simulation on hourly wind data and indeed followed a free fuzzy logic (FFL) design for prediction. The stable chaotic attractors do not influence the time series prediction. Jafri [16, 21], also studied the rule based on fuzzy logic time series prediction, feed forward back propagation neural networks (FFBPNN) and artificial neuro fuzzy information system (ANFIS). Her ARIMA and SARIMA (seasonal auto regressive integrated moving average) modeling techniques and statistical tests [16, 21], unraveled many hidden information with particular emphasis to aperiodic stable chaotic attractors. Lalarukh and Jafri [17] an ARMA process on hourly global radiation data, performed stochastic modeling through MTM (Markov Transition Matrix) and generated synthetic sequences of hourly global solar radiation. They found MTM approached relatively better as a simulator compared to ARMA (auto regressive moving average). But, their

analysis for ARMA process to simulate and forecast hourly averaged wind speed yielded good results [18]. We observed with critical analysis [1,4,6,14], [18-20] and with experimental data modeling and simulation [15,18, 21] that the aperiodic attractors (usually with sparse time delays) produced stable chaos whereas the periodic attractors the unstable chaos in any nonlinear dynamical systems. Jafri [22] suggested methods for Kolmogorov-Sinai disorder in terms of “point stochastic process” and found that this stochastic process can be measured with a positive Lyapunov exponent. The point stochastic process deals with discrete, independent and identically distributed random data. The non-stationary stochastic process other than point stochastic process in dynamical systems depend on negative Lyapunov exponents. The positive Lyapunov exponents are preferably found in point stochastic processes [2,3,23,24] where disorder introduces chaos in dynamical systems. Such chaotic dynamical system have been studied by real valued time series for random process [6,15,18, 21,22,17,18], Jafri [24] also studied disorder in a stochastic system in the form of Fisher information matrix in probability function, using the partition in the family of random variables for their corresponding degree of randomness for one-step prediction in time series, as a consequence of which, we obtained the disjoint sets of random variables. There is a literal difference between fractional exponents of Heaviside step function and the Lyapunov exponents. The Lyapunov exponents define attractors both in arbitrary positive and negative directions (hence positive and negative Lyapunov exponents) while the fractional exponents of exponential function in the denominator for the Heaviside step function show fragmented wavelet or wave fronts (piecewise step function) and do not represent attractors. With this limitation, the Fourier Transform is changed in to Laplace transform. Using the generalized definitions of Heaviside step function. By Bracewell [25], we observe that  $H(x)$  changes between 1 and 1/2 for  $x \geq 0$  and changes between 0 and 1/2 for  $x \leq 0$ .

Few of the expressions of the Heaviside step function [25] are written to reflect our conjectures true in the above paragraph.

$$H_n(x) = \lim_{t \rightarrow 0} \begin{cases} \frac{1}{2} e^{x/t}, & \text{if } x \leq 0 \\ 1 - \frac{1}{2} e^{-x/t}, & \text{if } x \geq 0 \end{cases} \quad (2)$$

where  $x/t$  is a fractional exponent. The aperiodic stable chaos differs from periodic unstable chaos [20]. The unstable periodic chaos [13], [14], [20] is associated with multiple iterations and for each iterations, periodicity is maintained whereas the stable aperiodic chaos [15] follow a singlet iteration (closed loop).

We shall consider the time-frequency representation of Fractional Fourier Transform (FRFT), [26]. Such as the Wigner distribution (WD), Wiener Space (WS), the ambiguity function, the short time Fourier transform (STFT) and the Spectrograms used in speech processing, radar, image rotations, confocal microscopy, etc. with rotation vector  $\alpha$  to define the aperiodic stable chaos with Lyapunov exponents in time series.

## 2. Results and Discussions

Almeida [26] define the kernel of FRFT in time frequency plane, i.e.,

$$K_{\alpha}(t, u) = \begin{cases} \sqrt{\frac{1-j \cot \alpha}{2\pi}} e^{j\frac{t^2+u^2}{2} \cot \alpha - jut \csc \alpha} & \text{if } \alpha \neq \pi \\ \delta(t - u) & \text{if } \alpha = 2n\pi \\ \delta(t + u) & \text{if } \alpha = (2n - 1)\pi \end{cases} \quad (3)$$

where  $n$  is an integer. With  $\alpha = \frac{\pi}{2}$ , the kernel in eq (3) coincides with the kernel of the Fourier transform (FT).

Almeida [26] redefined the FRFT of a function  $x(t)$ , with an angle  $\alpha$ , which is define in eq (4)

$$\mathcal{F}_{\alpha}[x(t)] = X_{\alpha}(u) = \int_{-\infty}^{\infty} x(t) K_{\alpha}(t, u) dt$$

$$= \begin{cases} \sqrt{\frac{1-j \cot \alpha}{2\pi}} e^{j\frac{u^2}{2} \cot \alpha} \int_{-\infty}^{\infty} x(t) e^{j\frac{t^2}{2} \cot \alpha - jut \csc \alpha} dt & \text{if } \alpha \text{ is not a multiple of } \pi \\ x(t) & \text{if } \alpha \text{ is not a multiple of } 2\pi \\ x(-t) & \text{if } \alpha + \pi \text{ is not a multiple of } 2\pi \end{cases} \quad (4)$$

Where  $u \equiv \omega$  (the angular frequency). We assume FRFT equivalent to  $H_{n_f}(x) \exp\left(-\frac{x^2}{2}\right)$  and time periodic signal  $x = A \sin \omega_0$  as Gaussian and exponentially correlated.  $A$  is real constant in  $x(t)$  while  $H_{n_f}(x)$  is a Hermite polynomial for  $n_f$  exponents (usually termed as Lyapunov exponents). The exponent of  $X_{\alpha}(u \equiv \omega)$  in eq (4) is  $\alpha$ . We assume the exponent  $\alpha$  of  $X_{\alpha}(u)$  to become Lyapunov exponents

$$X_{\alpha}(u \equiv \omega) \sim H_{n_f}(x) \exp\left(-\frac{x^2}{2}\right) \quad (5)$$

where  $\alpha = a \frac{\pi}{2}$ . Using our recent results on how fractional charge on an electron in the momentum space is quantized? [27], we find

$$H_{n_f}(\xi) = 2^{n_f} \quad (6)$$

Where  $H_{n_f}(\xi)$  are polynomials with  $n_f$ , Lyapunov exponent of 2. Eq (6) is consistent with the following definition of Hermite polynomials:

$$H_{n_f}(\xi) = (-1)^{n_f} e^{\xi^2} \frac{d^{n_f}}{d\xi^{n_f}} e^{-\xi^2} \quad (7)$$

$$H_{n_f}(\xi) = (-1)^{n_f} e^{\xi^2} \left(\xi - \frac{d}{d\xi}\right)^{n_f} e^{-\xi^2} \quad (8)$$

Eqs (7) and (8) show fractional exponents of  $\frac{d}{d\xi}$  and  $\left(\xi - \frac{d}{d\xi}\right)$ . The fractional exponents,  $n_f$  can be envisaged

with Lyapunov exponents for attractors. The Lyapunov exponents are both positive and negative fractional exponents. Hence, eq(6) is modified as

$$H_{n_f}(\xi) = 2^{\pm n_f} \tag{9}$$

With the Hermite generating function,

$$G(\xi, s) = e^{-s^2+2s\xi} \sum_{n=0}^{\infty} \frac{H_{n_f+1}(\xi)s^n}{n!} \tag{10}$$

and for  $n = 0$ , we can prove that the Hermite polynomials satisfy the recursion relations:

$$\begin{cases} H_{n_f+1}(\xi) - 2\xi H_{n_f} + H_{n_f-1}(\xi) = 0 \\ \frac{d}{d\xi} H_{n_f}(\xi) = 2n_f H_{n_f-1}(\xi) \end{cases} \tag{11}$$

Eq(10) shows that if the function  $e^{-s^2+2s\xi}$  is expanded in a power series in  $s$ , the coefficients of successive powers of  $s$  are just  $\frac{1}{n!}$  times the Hermite polynomials,  $H_{n_f}$ .

Using eq(11) for each of the fractional exponents,  $n_f$  on  $\lambda_f = 2n_{f+1}$  for  $E_{n_f} = (n_f + \frac{1}{2})\hbar\omega$ , there is

only one physically acceptable solution for energy profile of the attractor

$$\psi_{n_f}(\xi) = N_{n_f} e^{-\alpha^2 \frac{\xi^2}{2}} H_{n_f}(\xi) \tag{12}$$

where  $\xi \equiv x(t)$  and  $s \equiv \omega$  this implies that  $t \sim \alpha\xi$ . Considering the Hermite generating function eq(10) and equating the  $t$  coefficient of equal parts of  $s \equiv \omega$  and  $t \sim \alpha\xi$ , the normalized eigen functions for Lyapunov exponents are given by

$$\psi_{n_f}(x(t)) = \left(\frac{\alpha}{\sqrt{\pi} 2^{\pm n_f} n!}\right)^{\frac{1}{2}} e^{-\frac{\alpha^2 x^2(t)}{2}} H_{n_f}(\alpha\xi) \tag{13}$$

For Lyapunov exponents,  $\pm n_f$ ,  $\alpha = \omega_0 t$ ,  $n! = 0! = 1$  and  $x(t) = A \sin \omega_0 t$  as Gaussian, using eq(9) in eq(13). Eq (13) is modified as

$$\psi_{n_f}(A \sin \omega_0 t) = \left(\frac{\omega_0 t}{\sqrt{\pi} 2^{\pm n_f}}\right)^{\frac{1}{2}} e^{-\frac{\alpha^2 (A \sin \omega_0 t)^2}{2}} \times 2^{\pm n_f} = \left(\frac{\omega_0 t}{\sqrt{\pi} 2^{\pm n_f}}\right)^{\frac{1}{2}} 2^{\frac{\pm n_f}{2}} e^{-\alpha^2 \left(\frac{A \sin \omega_0 t}{\sqrt{2}}\right)^2} \tag{14}$$

the term  $e^{-\frac{\alpha^2 (A \sin \omega_0 t)^2}{2}} \sim e^{-\left(\frac{\omega_0 t A \sin \omega_0 t}{\sqrt{2}}\right)^2}$  is a convergent series and tends to unity in eq (14). With  $\sin \omega_0 t \sim \omega_0 t = \alpha$ , eq (14) becomes

$$\psi_{n_f}(\omega_o t) = \left(\frac{\omega_o t}{\sqrt{\pi}}\right)^{\frac{1}{2}} 2^{\frac{\pm n_f}{2}} \quad (15)$$

In a time series following aperiodic stable chaos (stationary stochastic process), the coefficient  $A$  of  $\sin \omega_o t$  is a real constant for an attractor with frequency  $\omega_o$ . At any time in a time series in the plane  $(t, \omega)$  [Wiener space], i.e.,  $t = t_{att}$  (time of flight for an attractor), a phase change of  $\omega_o t_{att}$  will cause a change in energy profile and hence an aperiodic stable chaotic attractors will be produced on a time series. The coefficient  $A$ , being a real constant, can be quantized with simulation. Equation (15) can be modified as

$$\psi_{\pm n_f}(\pm \omega_o t_{att}) = \left(\frac{\omega_o t}{\sqrt{\pi}}\right)^{\frac{1}{2}} 2^{\frac{\pm n_f}{2}} \quad (16)$$

$$\text{where } \frac{\pi}{2} < \omega_o t \leq 2\pi \text{ and } 0 \leq \omega_o t < \frac{\pi}{2}, \text{ i.e., } \omega_o t = \alpha \neq \frac{\pi}{2}$$

Eq(16) represents shape of profile for aperiodic stable attractors with Lyapunov exponents,  $\pm n_f$ . The periodic unstable attractors at  $\alpha = \omega_o t = \frac{\pi}{2}$  will be produced following the FT. Eq (16) obeys WD and WS in  $(t, \omega)$  plane.

All attractors whether periodic or aperiodic are, in fact, the manifestation of Lyapunov exponents.

### 3. Conclusion

The shape of energy profile for aperiodic stable attractors with Lyapunov exponents  $\pm n_f$  is determined by using fractional Fourier transform. The formula aperiodic stable attractors eq (16) is established. With  $\omega_o t = \frac{\pi}{2}$ , it is found that the energy profile exists for periodic unstable attractors at different values of Lyapunov exponents,  $n_f$ . Critical analysis about chaos is revisited with emphasis on time series modeling.

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