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Some Solution Methods for Lane-Emden Differential Equation when the Polytropic Index is Three

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Abstract

The Lane-Emden equation is one of the widely used and challenging equations in nonlinear dynamics. It is a central equation in the theory of steller structures. In this paper, Lane-Emden equation when $n = 3$ has been solved using few different methods. Pade approximant has also been calculated of the obtained solution. The results obtained using semi-analytical methods are compared with the solutions obtained using well-known numerical methods namely Runge-Kutta's fourth order method and ODE 45 graphically.

Keywords: Adomian decomposition method; Differential transform method; Pade approximation; Lane-Emden differential equations.

1. Introduction

In the study of steller structure, the Lane-Emden equation

$$y''(x) + \frac{2}{x}y'(x) + y^n = 0 \qquad (1)$$

with initial conditions $y(0) = 1$ and $y'(0) = 0$, is a nonlinear ordinary differential equation of second order which models the thermal behavior of a spherical cloud of gas under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. Here n ia s constant, which is called the polytropic index.

In astrophysics, the Lane–Emden equation is a dimensionless form of Poisson's equation for the gravitational potential of a Newtonian self-gravitating, spherically symmetric, polytrophic fluid.

The Lane-Emden equation is proposed by astrophysicists Jonathan Homer Lane [1] who first published it in 1870 and studied in detail by Robert Emden.

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The exact solutions have been found for $n = 0,1$ and 5. Solutions for all other values should be obtained numerically. For $n = 3$ of the equation (1) which gives

$$y''(x) + \frac{2}{x}y'(x) + y^3 = 0 \qquad (2)$$

with initial conditions $y(0) = 1$ and $y'(0) = 0$ is called the Eddington's approximation and is an interesting case since it corresponds to a useful approximation of the sun.

A number of techniques have been used to solve Lane-Emden equations. The Differential transform method has been applied to find the solution for Lane-Emden equation with polytropic index n in [2]. A numerical solution for differential equations of Lane-Emden type has been found in [3]. Perturbative solution to the Lane-Emden equation with eigenvalue approach has been considered in [4].A special second order Lane-Emden differential equation has been solved by He's variational iteration, Adomian decomposition method , Homotopy analysis method, Homotopy perturbation method and Finite difference method in [5]. An efficient numerical method for solving Lane–Emden type equations was applied using Bernstein polynomials in [6]. An iterative method has been introduced for solving two special cases of Lane-Emden type equation by [7].

Authors in [2] and [6] have solved Lane-Emden type equations using differential transform method, Adomian method and modified Adomian decomposition method respectively. But they have not considered about this special case of Lane-Emden equation which is a very important equation in Steller structures in Physics.

In this paper, Lane-Emden equation for $n = 3$ is solved using few different semi-analytical and numerical methods. Since the exact solution of the above equation does not exist, we solve the equation using well-known numerical method Runge-Kutta's fourth order method and ODE 45 using MATLAB software. The comparisons of the results obtained using different methods will be done graphically with the solutions obtained using above numerical methods.

2. Material and the Method

In this section, the solution for the Lane-Emden equation has been obtained using the methods Adomian Decomposition method (ADM), a Modified Adomian Decomposition method and the Differential Transform Method (DTM).

2.1 The Solution with Adomian Decomposition Method

The Adomian decomposition method [8], also known as the inverse operator method, is a mathematical method for solving linear and nonlinear mathematical physics equations; it was proposed by George Adomian.

In standard Adomian decomposition method, we define linear operator L and its inverse operator by

$$L(\cdot) = x^{-2}\frac{d}{dx}\left(x^2\frac{d}{dx}\right)(\cdot) \qquad (3)$$

$$L^{-1}(\cdot) = \int_0^x x^{-2} \int_0^x x^2(\cdot)\, dx\, dx \quad (4)$$

Using the operators given in (3) and (4), the equation (2) can be written in the form

$$Ly = -y^3 \quad (5)$$

Then solution of the equation (2) takes the form

$$y(x) = y(0) - L^{-1}(y^3) \quad (6)$$

Taking $y(x) = \sum_{n=0}^{\infty} y_n$ and writing the nonlinear term y^3 using Adomian polynomials as

$y^3 = \sum_{n=0}^{\infty} A_n$, we have

$$\sum_{n=0}^{\infty} y_n = 1 - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \quad (7)$$

or $$y_0 = 1 \quad (8)$$

$$y_{n+1} = -L^{-1}(A_n),\ n = 0,1,2,\dots \quad (9)$$

Using the definition of the inverse operator as in (4), we have

$$y_{n+1} = -\int_0^x x^{-2} \int_0^x x^2(A_n)\, dx\, dx \quad (10)$$

It is convenient to list the first few Adomian polynomials. Following [9] and after some calculations, the first few Adomian polynomials take the form

$$A_0 = 1$$

$$A_1 = -\frac{x^2}{6}$$

$$A_2 = \frac{19}{120}x^4$$

$$A_3 = \frac{357}{5040}x^6$$

and so on. Now inserting these Adomian polynomials to equation (8) and (9) gives

$$y_0 = 1,$$

$$y_1 = -L^{-1}(A_0) = -\frac{1}{6}x^2,$$

$$y_2 = -L^{-1}(A_1) = \frac{1}{40}x^4$$

$$y_3 = -L^{-1}(A_2) = -\frac{19}{5040}x^6$$

$$y_4 = -L^{-1}(A_3) = \frac{-357}{362880}\ x^8$$

Therefore, substituting above polynomials to the equation (7), the solution of the Lane-Emden equation when $n = 3$ can be written in the form

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{40}x^4 - \frac{19}{5040}x^6 - \frac{357}{362880}x^8 + \cdots \quad (11)$$

2.2 The Solution Using a Modified Adomian Decomposition Method

In this method, Adomian decomposition method was applied in a straightforward manner with a new choice for the differential operator [9].

The differential operator L_1 together with its corresponding inverse integral operator L_1^{-1} are defined as

$$L_1(\cdot) = \frac{1}{x}\frac{d^2}{dx^2}x(\cdot) \quad (12)$$

and
$$L_1^{-1}(\cdot) = \frac{1}{x}\int_0^x\int_0^x x(\cdot)dxdx. \quad (13)$$

Now equation (2) takes the form

$$L_1 y = -y^3 \quad (14)$$

As in the section 2.1, the solution of the equation (2) can be written in the form

$$y(x) = y(0) - L^{-1}(y^3) \quad (15)$$

Taking $y(x) = \sum_{n=0}^{\infty} y_n$ and writing the nonlinear term y^3 using Adomian polynomials as $y^3 = \sum_{n=0}^{\infty} A_n$, we have

$$\sum_{n=0}^{\infty} y_n = 1 - L_1^{-1}\sum_{n=0}^{\infty} A_n \quad (16)$$

$$y_0 = 1 \quad (17)$$

$$y_{n+1} = -L_1^{-1}(A_n), n = 0,1,2, \dots \quad (18)$$

Using the definition of the inverse operator as in (13), we have

$$y_{n+1} = -\frac{1}{x}\int_0^x\int_0^x x(A_n)dxdx \quad (19)$$

As in the section 2.1, the first few Adomian polynomials take the form

$$A_0 = 1$$

$$A_1 = -\frac{x^2}{6}$$

$$A_2 = \frac{19}{120}x^4$$

$$A_3 = \frac{357}{5040}x^6$$

and so on. Now inserting these Adomian polynomials to equation (8) and (9) gives

$$y_0 = 1,$$

$$y_1 = -L^{-1}(A_0) = -\frac{1}{6}x^2,$$

$$y_2 = -L^{-1}(A_1) = \frac{1}{40}x^4$$

$$y_3 = -L^{-1}(A_2) = -\frac{19}{5040}x^6$$

$$y_4 = -L^{-1}(A_3) = \frac{-357}{362880}\ x^8$$

Therefore, substituting above polynomials to the equation (15), the solution of the Lane-Emden equation when $n = 3$ can be written in the form

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{40}x^4 - \frac{19}{5040}x^6 - \frac{357}{362880}x^8 + \cdots \tag{20}$$

2.3 The Solution Using Differential Transform Method

The differential transform technique is one of the semi- analytical numerical methods for ordinary and partial differential equations that use the form of polynomials as approximations of the exact solutions that are sufficiently differentiable which is first introduced by Zhou [10] in 1986.The method is an analytical an numerical method for solving a wide variety of differential equations and usually gets the solution in series form.

Differential transform of a function $f(x)$ is defined by

$$F(k) = \frac{1}{k}\left[\frac{d^k f}{dx^k}\right]_{x=0} \tag{21}$$

where $f(x)$ is the original function and $F(k)$ is the transformed function.

Inverse differential transform is defined by

$$f(x) = \sum_{k=0}^{\infty} \left[\frac{d^k f}{dx^k}\right]_{x=0} \frac{x^k}{k!} \tag{22}$$

A few basic results which have been proved and can be found in the literature are given by following table.

Table 1: Basic results of Differential Transform Method

The function $f(x)$	**The transformed function $F(k)$**
$g(x) \pm h(x)$	$G(k) \pm H(k)$
$cg(x)$	$cG(k)$
$\frac{d^n g(x)}{dx^n}$	$\frac{(k+n)!}{k!} G(k+n)$
$g(x)h(x)$	$\sum_{l=0}^{k} G(k)H(k-l)$
x^n	$\delta(k-n)$
$\int_0^x g(t)dt$	$\frac{G(k-1)}{k} \quad k \geq 1$
$u(x)v(x)w(x)$	$\sum_{r=0}^{k} \sum_{s=0}^{k-r} \sum_{m=0}^{k-r-s} U(r)V(s)W(m)$

where $G(k), H(k), U(r), V(s)$ and $W(m)$ are the transformed functions of $g(x), h(x)$, $u(x)$

$v(x)$ and $w(x)$, respectively and $\delta(r)$ is the Kronecker delta function.

After rewriting the equation (2), we get

$$xy''(x) + 2y'(x) + xy^3 = 0 \tag{23}$$

Applying the results given in the table 01 to the equation (3), we have

$$\sum_{l=0}^{k} \delta(l-1)\,(k+2-l)Y(k+2-l) + 2(k+1)Y(k+1) +$$

$$\sum_{r=0}^{k} \sum_{s=0}^{k-r} \sum_{m=0}^{k-r-s} \delta(r-1)Y(s)Y(m)Y(k-r-s-m) = 0 \tag{24}$$

where $Y(k)$ is the transformed function of $y(x)$.

Transformed initial conditions gives

$$Y(0) = 1 \qquad \text{and} \qquad Y(1) = 0. \tag{25}$$

Using the results in (25) with the equation (24) and solving the iterative equations, gives

$Y(2) = -\frac{1}{6}$,

$Y(4) = \frac{1}{40}$,

$Y(6) = \frac{-19}{5040}$,

$Y(8) = \frac{619}{1088640}$.

and $\quad Y(3) = Y(5) = Y(7) = 0$.

Now using the definition of inverse differential transform (22), the solution takes the form,

$$y(x) = 1 - \frac{1}{6}x^2 + \frac{1}{40}x^4 - \frac{19}{5040}x^6 + \frac{619}{1088640}x^8 - \cdots . \qquad (26)$$

Here we can see from equations (11), (20) and (26) that the solution of the Lane-Emden equation when $n = 3$ are the same for all the methods we used.

Since the Padé approximant [11] often gives better approximation of the function than truncating its Taylor series and it may still work where the Taylor series does not converge, to increase the accuracy of the solution, Pade approximant $[4/4]$ have been used.

After calculating Pade approximant $[4/4]$, the solution takes the form

$$y(x) = \frac{1 - 0.009259259233694x^2 - 0.000242504409596343 8x^4}{1 + 0.157407407432973x^2 + 0.000992063495899x^4} \qquad (27)$$

Since there are no exact solution for this equation, in the next section, Runge-Kutta's fourth order method and ODE 45 will be used to find the solution of the equation (2).

2.4 The Solution Using Runge-Kutta's Fourth Order Method

Runge-Kutta method was formulated by the German mathematicians, Runge and Kutta. The 4th order Runge-Kutta method approximates the curve of the function f between two points by a 4th degree polynomial. It requires 4 slope estimates at each step. Geometrically, we can visualize the curve between two points as part of a 4th degree polynomial.

Equations for the fourth order Runge-Kutta's method for a single ordinary differential equation are given by

$$y_{n+1}(x) = y_n(x) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) , \qquad n = 0,1,2,3 \ldots \qquad (28)$$

where $\quad k_1 = h\, f(x, y)$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right)$$

$$k_4 = hf(x + h, y + k_3).$$

The table of obtained values the $y(x)$ of equation (2) for different values of x by using Runge-Kutta's fourth order method will be given in section 3.

2.5 The Solution Using ODE 45 Method

ODE 45 is a solver of ordinary differential equations which is using in MATLAB software. It is constructed with Runge-Kutta's method . ODE 45 can be used to obtain solutions with high accuracy.

The table of obtained values for the $y(x)$ of equation (2) for different values of x by ODE 45 method will be given in results section.

3. Results

The values for the solution of equation (2) using different methods have been tabulated as follows:

Table 2: Solutions obtained using ADM or DTM with Pade approximation, Runge-Kutta's fourth order method and ODE 45 method

x	ADM and DTM with Pade Approximation	Runge-Kutta's fourth order	ODE 45
0.0	1.000000000000000	1.000000000000000	1.000000000000000
0.5	0.95983906946246	0.882334391276042	0.959829596023597
1.0	0.855057569702160	0.818324571511985	0.855038804145293
1.5	0.719501847543459	0.708922682687295	0.719520205360974
2.0	0.582851018220845	0.588530418166383	0.582887896307385
2.5	0.461129448151790	0.475572766200929	0.461230453835357
3.0	0.359237187128084	0.377621800786575	0.359326962683592
3.5	0.276292569751981	0.295897586463492	0.276388576584968
4.0	0.209350163630244	0.228791800654301	0.209413356464647
4.5	0.155204664160547	0.173852507267953	0.155205230683796
5.0	0.111058197915109	0.128672559758217	0.110959356556518
5.5	0.0746697821433852	0.091198640909757	0.074426172397138
6.0	0.044311377259109	0.059784221403022	0.043879722409806
6.5	0.018671487058125	0.033147611317745	0.018007888556991
7.0	-0.003240837308288	0.010305351987701	-0.004169636604097
7.5	-0.022175774117545	-0.009492133113330	-0.023390625163479
8.0	-0.038704415668096	-0.026814393625509	-0.040204727092085
8.5	-0.053264924046445	-0.042093399286705	-0.055024532465237
9.0	-0.066196562308791	-0.055656119406371	-0.068156949384751
9.5	-0.077764651766161	-0.067749292650195	-0.079830913010358
10.0	-0.088178871397516	-0.078558271426851	-.090215526101679

For the sake of comparison, the all three graphs have been plotted in the same plot using MATLAB software.

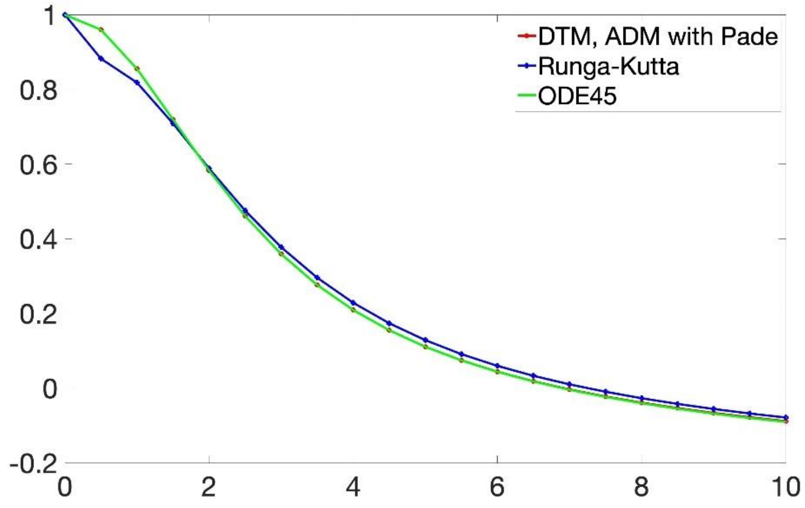

Figure 1: The solutions of Lane-Emden equation when $n = 3$ using different methods

4. Discussion

In section 3.1, using standard Adomian decomposition method, a solution has been obtained.

In section 3.2, changing the differential operator, a solution has been obtained. The solutions obtained using standard Adomian method as well as a new method which has a new form of differential operator takes the same form for the range of values we considered.

The Differential transform method has been used to solve Lane-Emden equation when $n = 3$ in section 3.3. The terms of the infinite series has been obtained by an Iterative relation using MATLAB software. To increase the accuracy of the solution , Pade approximation[4/4] of the solution has been considered. The solution obtained here, also takes the same form of the solution we have in the two pervious sections.

So for the considered range of values, the solutions obtained for the Lane-Emden equation when $n = 3$ using the above three different methods take the same form.

Since there is no exact solution for Lane-Emden equation when $n = 3$, to check the accuracy of the obtained solution, we solve the equation using well known numerical methods Runge-Kutta's fourth order method and ODE 45 method with the help of MATLAB software.

To the clarification of the results, the solutions obtained using all methods have been graphed in the same plot in Figure 1. According to the plot, all the graphs are going together except some points which means that the solutions obtained using few different semi-analytical and numerical are agreed for the range of values we considered.

In this study, only a special type of Lane-Emden equation has been taken into account and solved by different methods. The constraints we met in this study are as follows.

The solutions for the Lane -Emden equation for $n = 3$ has been calculated for a small range of values for . When the range of values is larger, the solutions obtained by methods we used will not be gone with the solutions obtained by the well-known method such as Runge-Kutta's fourth order method.

Also when the number of terms of the infinite series is increasing , calculations will be more harder than considering only first few terms. But for the sake of the accuracy, we have to increase the number of terms of the infinite series.

5. Conclusion

Since Runge-Kutta's method is a numerical method for solving ordinary differential equations with high accuracy, the solution obtained by using Adomian decomposition method, a modified-Adomian decomposition method and differential transform method is a good approximation for the Lane-Emden equation with $n = 3$.

Therefore, with the proper investment of Adomian decomposition method, it is possible to attain an good approximation to Lane-Emden equation with $n = 3$. The solution has been obtained in the form of a series with comparably easy calculations and rapid convergence in the interval considered.

A different form of inverse integral operator based on the standard Adomian decomposition method considered and effectively applied them to Lane-Emden equation considered. The solution was the same with the standard ADM and it can be seen that using this method also, a good approximation can be obtained.

According to the Figure 1 in the section 3, differential transform method is a very fact convergent, precise and time saving method for solving Lane-Emden equation considered.

Thus, differential transform is an important tool in handling highly nonlinear differential equations with a minimum size of computations. Comparing with the differential transform method, Adomian decomposition method and modified Adomian decomposition method need large quantity of computations.

Although the solutions agree with the numerical solutions in the small interval considered, generally for large x values, the solution will be diverging. Therefore, as a solution for this problem the differential transform method with Laplace transform and Pade approximation will be considered.as a future work.

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