Three-dimensional flow of a second grade fluid along an infinite horizontal plane wall with periodic suction

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Abstract: In this paper, three-dimensional flow of a second grade fluid along a horizontal infinite plate which is subjected to a transverse sinusoidal suction velocity distribution is studied. Due to variable suction velocity distribution the flow becomes three-dimensional and for constant suction the problem becomes two-dimensional. The free stream velocity is uniform and for small perturbation approximation, analytic technique is applied to obtain the expressions for velocity field and components of skin friction. The effect of second-grade parameter, Reynolds number and suction parameter on the velocity in the direction of main flow and on the stress components is investigated with the help of graphs. The existence of backflow is observed and it is noted that the Reynolds number and suction parameter are controlling parameters for the backflow.

Keywords: Differential type fluids; Three-dimensional flows; Periodic suction; Regular perturbation method; Series solutions

1 Introduction

The research area of laminar flow control has received attention of many investigators in recent years and this research area is continuously growing. One of the important applications of laminar flow is the calculation of friction drag of bodies in a flow i.e. the drag of a plate at zero incidence, an airfoil and the friction drag. The main purpose is to reduce drag and hence to improve the vehicle power by a considerable amount. The transition from laminar to turbulent flow which results the drag coefficient to increase, may be prevented or deferred by the suction of fluid and heat transfer from boundary layer to the wall [1]. Gersten et al. [2] have investigated the effect of transverse sinusoidal suction velocity on flow and heat transfer along an infinite porous wall. Singh et al. [3] investigated the flow of viscous incompressible fluid along an infinite porous plate when the transverse sinusoidal suction velocity distribution fluctuating with time is applied. Also Singh et al. [4] have examined the effect of buoyancy forces on three-dimensional flow and heat transfer along with porous vertical plate. Singh [5] extended this idea by applying transverse sinusoidal suction velocity in the presence of viscous dissipative heat. Singh et al. [6] studied the effects of magnetic field on the three-dimensional flow past a porous plate. Transient three-dimensional viscous fluid flow along a porous plate has been studied by Singh et al. [7] while Guria et al. [8] have presented hydrodynamics effect on the three-dimensional flow past a vertical porous plate. Gupta et al. [9] observed MHD effect on the three-dimensional flow past a porous plate.

All the above problems have been investigated in viscous fluid. Although the Navier–Stokes equations can manage the flows of viscous fluids but such equations are not adequate to describe the properties of non-Newtonian fluids. Other than viscous fluids there is not a single model which can describe the properties of all non-Newtonian fluids. Therefore, several constitutive relationships of non-Newtonian fluids have been proposed. Generally, non-Newtonian fluids have been classified into three main categories namely the differential, rate and integral types. Second-grade fluid is the simplest subclass of differential type fluids.

The aim of present study is to discuss three-dimensional flow of a second-grade fluid along a plane wall which is subjected to the sinusoidally varying velocity distribution. A constant suction velocity at the wall leads to two-dimensional asymptotic suction solution [10], however, due to variation of suction velocity in transverse direction on wall the problem becomes three-dimensional. The regular perturbation method is employed for the solution of the present problem. The results obtained are evaluated for different values of dimensionless parameters such as non-Newtonian elastic parameter K, Reynolds number R_e and suction parameter α. The article is organized as follows: Section 2 presents the problem description, Section 3 describes the formulation of the problem, Section 4 gives perturbation solutions, Section 5 incorporates results and discussion, while Section 6 includes conclusion.

2 Description of the problem

Consider the three-dimensional laminar flow of an incompressible second-grade fluid past an infinite plane wall. A Cartesian coordinate system with the wall lying on xz-plane and the y-axis normal to it is introduced. A suction velocity distribution [2] consisting of a basic steady distribution ($v_0 > 0$) with a superimposed weak transversely varying distribution

$\epsilon v_0 \cos\left(\pi \frac{z}{l}\right)$, where l denotes the wave length of the periodic suction velocity distribution and ϵ the amplitude of the suction velocity variation, is taken. Thus,

$$v(z) = -v_0 \left(1 + \varepsilon \cos \pi \frac{z}{l}\right). \tag{1}$$

The constant suction velocity v_0 at the wall leads to the well known two-dimensional asymptotic suction solution [10] while varying suction velocity distributions lead to a cross flow and hence to a three-dimensional flow over the surface. All the physical quantities will be independent of x because of the infinite length of the wall in the x-direction, of course, the flow remains three-dimensional due to variation of suction velocity.

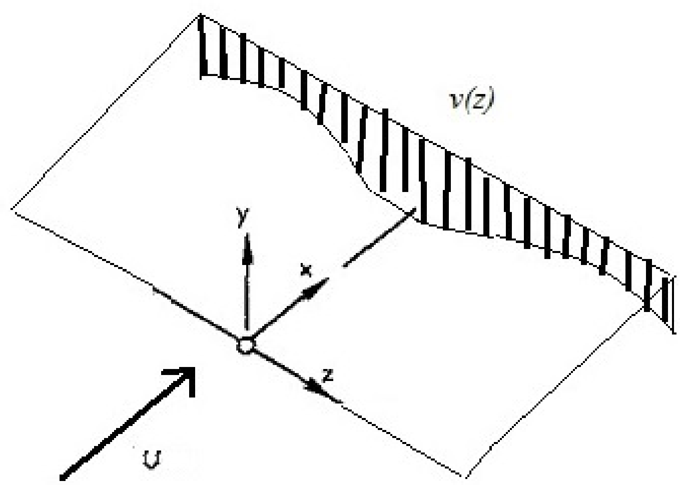


Fig.1 Geometry of the problem

3 Formulation of the problem

Consider the three-dimensional laminar flow of an incompressible second-grade fluid past an infinite wall , with the x-axis on the wall parallel to the direction of flow. We applied suction velocity distribution [2] of the form $v(z) = -v_0(1+\epsilon \cos \pi \frac{z}{l})$, where $(v_0 > 0)$, l and ϵ are the suction velocity, wave length of the periodic suction velocity distribution and amplitude of the suction velocity distribution. As we have considered asymptotic flow, therefore velocity field is independent of x. In case of constant suction we have well-known two-dimensional asymptotic suction solution and variable suction velocity distribution leads to cross-flow which results in three-dimensional flow.

The constitutive expression for second-grade fluid model is

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\mu}\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2}$$

in which p, $\mathbf{I}$, μ, $\alpha_i\,(i = 1, 2)$ denote the pressure, the identity tensor, the dynamic viscosity and material constants respectively. The Rivlin-Ericksen tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined as

$$\begin{aligned} A_1 &= L+L^T, \\ \mathbf{A}_2 &= \frac{d\mathbf{A}_1}{dt}+\mathbf{A}_1 L + L^T\mathbf{A}_1, \\ L &= \nabla\mathbf{V}, \end{aligned} \tag{3}$$

where ∇ is the operator, $\mathbf{V}$ is the velocity field. For the model (2) required to be compatible with the thermodynamics in the sense that all motions satisfy the Clasius-Duhen inequality and assumption that the specific Helmholtz free energy is a minimum in equilibrium, then the material parameters must meet the following conditions [11]

$$\mu \geq 0, \alpha_1 \geq 0 \text{ and } \alpha_1 + \alpha_2 = 0. \tag{4}$$

The laws of conservation of mass and momentum for the present flow problem are given by

$$\frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0, \tag{5}$$

$$\begin{aligned} \rho\left(v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}\right) &= \mu\left[\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}}\right] \\ &+\alpha_1\left[v^*\frac{\partial^3 u^*}{\partial y^{*3}} + w^*\frac{\partial^3 u^*}{\partial y^{*2}\partial z^*} + v^*\frac{\partial^3 u^*}{\partial y^*\partial z^{*2}} + w^*\frac{\partial^3 u^*}{\partial z^{*3}}\right], \end{aligned} \tag{6}$$

$$\begin{aligned} \rho\left(v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}\right) &= -\frac{\partial p^*}{\partial y} + \mu\left[\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}}\right] \\ &+\alpha_1\left[\begin{array}{c} v^*\frac{\partial^3 v^*}{\partial y^{*3}} + w^*\frac{\partial^3 v^*}{\partial y^{*2}\partial z^*} + v^*\frac{\partial^3 v^*}{\partial y^*\partial z^{*2}} + w^*\frac{\partial^3 v^*}{\partial z^{*3}} + \frac{\partial v^*}{\partial z^*}\frac{\partial^2 v^*}{\partial y^*\partial z^*} \\ +\frac{\partial u^*}{\partial z^*}\frac{\partial^2 u^*}{\partial y^*\partial z^*} + 5\frac{\partial v^*}{\partial y^*}\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial v^*}{\partial z^*}\frac{\partial^2 w^*}{\partial y^{*2}} + 2\frac{\partial u^*}{\partial y^*}\frac{\partial^2 u^*}{\partial y^{*2}} \\ +2\frac{\partial w^*}{\partial y^*}\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial u^*}{\partial y^*}\frac{\partial^2 u^*}{\partial z^{*2}} + \frac{\partial v^*}{\partial y^*}\frac{\partial^2 v^*}{\partial z^{*2}} \end{array}\right] \end{aligned} \tag{7}$$

$$\begin{aligned} \rho\left(v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}\right) &= -\frac{\partial p^*}{\partial z^*} + \mu\left[\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}}\right] \\ &+\alpha_1\left[\begin{array}{c} w^*\frac{\partial^3 v^*}{\partial y^{*3}} + v^*\frac{\partial^3 w^*}{\partial y^{*3}} + v^*\frac{\partial^3 w^*}{\partial y^*\partial z^{*2}} + w^*\frac{\partial^3 w^*}{\partial z^{*3}} + \frac{\partial w^*}{\partial y^*}\frac{\partial^2 w^*}{\partial y^*\partial z^*} \\ +\frac{\partial u^*}{\partial y^*}\frac{\partial^2 u^*}{\partial y^*\partial z^*} + 5\frac{\partial w^*}{\partial z^*}\frac{\partial^2 w^*}{\partial z^{*2}} + \frac{\partial w^*}{\partial y^*}\frac{\partial^2 v^*}{\partial z^{*2}} + 2\frac{\partial u^*}{\partial z^*}\frac{\partial^2 u^*}{\partial z^{*2}} \\ +2\frac{\partial v^*}{\partial z^*}\frac{\partial^2 v^*}{\partial z^{*2}} + \frac{\partial u^*}{\partial z^*}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial w^*}{\partial z^*}\frac{\partial^2 w^*}{\partial y^{*2}} \end{array}\right] \end{aligned} \tag{8}$$

with the boundary conditions [2]

$$\begin{aligned} u^* &= 0,\ v^* = -v_0(1+\epsilon\cos\pi\frac{z^*}{l}),\ w^* = 0 \text{ at } y^* = 0, \\ u^* &= U,\ v^* = -v_0,\ w^* = 0,\ p^* = p^*_\infty \text{ as } y^* \to \infty, \end{aligned} \tag{9}$$

in which u^*, v^* and w^* denote the velocities in the x^*-, y^*- and z^*-directions, respectively.

We now introduce the following non-dimensional variables [9]:

$$y = \frac{y^*}{l},\ z = \frac{z^*}{l},\ u = \frac{u^*}{U},\ v = \frac{v^*}{U},\ w = \frac{w^*}{U},\ p = \frac{p^*}{\rho U^2}. \tag{10}$$

Then the Eqs. (5) – (9) become

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{11}$$

$$\begin{aligned} v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= \frac{1}{R_e}\left[\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \\ &\quad +K\left[v\frac{\partial^3 u}{\partial y^3} + w\frac{\partial^3 u}{\partial y^2\partial z} + v\frac{\partial^3 u}{\partial y\partial z^2} + w\frac{\partial^3 u}{\partial z^3}\right], \end{aligned} \tag{12}$$

$$\begin{aligned} v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + \frac{1}{R_e}\left[\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] \\ &\quad +K\left[\begin{array}{c} v\frac{\partial^3 v}{\partial y^3} + w\frac{\partial^3 v}{\partial y^2\partial z} + v\frac{\partial^3 v}{\partial y\partial z^2} + w\frac{\partial^3 v}{\partial z^3} + \frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial y\partial z} \\ +\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y\partial z} + 5\frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial y^2} + \frac{\partial v}{\partial z}\frac{\partial^2 w}{\partial y^2} + 2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} \\ +2\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial z^2} + \frac{\partial v}{\partial y}\frac{\partial^2 v}{\partial z^2} \end{array}\right], \end{aligned} \tag{13}$$

$$\begin{aligned} v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + \frac{1}{R_e}\left[\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right] \\ &\quad +K\left[\begin{array}{c} w\frac{\partial^3 v}{\partial y^3} + v\frac{\partial^3 w}{\partial y^3} + v\frac{\partial^3 w}{\partial y\partial z^2} + w\frac{\partial^3 w}{\partial z^3} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y\partial z} \\ +\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial z} + 5\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2} + \frac{\partial w}{\partial y}\frac{\partial^2 v}{\partial z^2} + 2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2} \\ +2\frac{\partial v}{\partial z}\frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y^2} + \frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial y^2} \end{array}\right], \end{aligned} \tag{14}$$

and the boundary conditions take forms

$$\begin{aligned} u &= 0,\ v = v(z) = -\alpha(1+\epsilon\cos\pi\frac{z}{l}),\ w = 0 \text{ at } y = 0, \\ u &= 1,\ v = -\alpha,\ w = 0, \text{ as } y \to \infty, \end{aligned} \tag{15}$$

where

$$R_e = \frac{Ul}{\nu},\ \alpha = \frac{v_0}{U},\ K = \frac{\alpha_1}{\rho l^2}. \tag{16}$$

4 Solution of the problem

Since ϵ is very small, therefore we assume solution in such a way

$$F = F_0 + \epsilon F_1 + \epsilon^2 F_2 + \cdots \tag{17}$$

where F stands for any of u, v, w and p. For $\epsilon = 0$, the problem becomes two-dimensional, so we have

$$K\alpha \frac{d^3 u_0}{dy^3} - \frac{1}{R_e}\left(\frac{d^2 u_0}{dy^2}\right) + \alpha \frac{du_0}{dy} = 0, \tag{18}$$

subject to boundary conditions

$$\begin{aligned} u_0 &= 0, \text{ at } y = 0, \\ u_0 &= 1 \text{ as } y \to \infty. \end{aligned} \tag{19}$$

The order of differential equation is increased from 2 to 3 due to presence of elasticity parameter. We are required three boundary conditions for unique solution of Eq. (18). To remove this difficulty we assume the solution of the form

$$u_0 = u_{00} + K u_{01} + O(K^2), \tag{20}$$

where K is very small parameter.

Using Eq. (20) in Eqs. (18)-(19) and comparing coefficients of $O(K^0)$ and $O(K)$, we get the following boundary value problems:

$$\begin{gathered} \frac{d^2 u_{00}}{dy^2} + \alpha R_e \frac{du_{00}}{dy} = 0, \\ u_{00}(0) = 0, \ u_{00}(\infty) = 1. \end{gathered} \tag{21}$$

$$\begin{gathered} \alpha R_e \frac{d^3 u_{00}}{dy^3} - \frac{d^2 u_{01}}{dy^2} - \alpha R_e \frac{du_{01}}{dy} = 0, \\ u_{01}(0) = 0, \ u_{01}(\infty) = 0. \end{gathered} \tag{22}$$

Solving the boundary value problems $(21) - (22)$ to obtain

$$u_{00}(y) = \left(1 - e^{-\alpha R_e y}\right), \tag{23}$$

$$u_{01}(y) = -\left(\alpha R_e\right)^3 y e^{-\alpha R y}. \tag{24}$$

Therefore, in view of Eqs. (23) and (24) , Eq. (20) yields

$$u_0(y) = 1 - e^{-\alpha R_e y} - K (\alpha R_e)^3 y e^{-\alpha R_e y}. \tag{25}$$

When $\epsilon \neq 0$, the solution of the problem is obtained by the perturbation method

$$u = u_0(y) + \epsilon u_1(y, z) + O(\epsilon^2), \tag{26}$$

$$v = v_0 + \epsilon v_1(y, z) + O(\epsilon^2), \tag{27}$$

$$w = w_0 + \epsilon w_1(y, z) + O(\epsilon^2). \tag{28}$$

Using Eqs. (26)-(28) into Eqs. (11)-(15) to obtain differential equations corresponding to first order terms

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0, \tag{29}$$

$$-\alpha \frac{\partial u_1}{\partial y} + v_1 \frac{\partial u_0}{\partial y} = \frac{1}{R_e} \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2} \right) + K \left(-\alpha \frac{\partial^3 u_1}{\partial y^3} - \alpha \frac{\partial^3 u_1}{\partial y \partial z^2} + v_1 \frac{\partial^3 u_0}{\partial y^3} \right), \tag{30}$$

$$-\alpha \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{1}{R_e} \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) - K\alpha \left(\frac{\partial^3 v_1}{\partial y^3} + \frac{\partial^3 v_1}{\partial y \partial z^2} \right), \tag{31}$$

$$-\alpha \frac{\partial w_1}{\partial y} = -\frac{\partial p_1}{\partial z} + \frac{1}{R_e} \left(\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2} \right) - K\alpha \left(\frac{\partial^3 w_1}{\partial y^3} + \frac{\partial^3 w_1}{\partial y \partial z^2} \right), \tag{32}$$

and the boundary conditions

$$\begin{aligned} u_1 &= 0,\ v_1 = -\alpha \cos \pi \frac{z}{l},\ w_1 = 0 \text{ at } y = 0, \\ u_1 &= 0,\ v_1 = 0,\ w_1 = 0,\ \text{as } y \to \infty. \end{aligned} \tag{33}$$

The set of linear differential equations $(29) - (33)$ describe the three-dimensional flow.

4.1 Cross flow Solution

In this section the set of cross-flow solutions $v_1(y, z)$, $w_1(y, z)$ and $p_1(y, z)$ are considered. This set of solution is independent of the main flow component u. The suction velocity consists of basic uniform distribution v_0 with a superimposed weak sinusoidal distribution $v_0 \varepsilon \cos(\pi z)$, therefore the velocity components $v_1(y, z)$, $w_1(y, z)$ and pressure $p_1(y, z)$ are also separated into main and small sinusoidal components. Therefore, assume the following forms for $v_1(y, z)$, $w_1(y, z)$ and $p_1(y, z)$:

$$v_1(y, z) = v_{11}(y) \cos \pi z, \tag{34}$$

$$w_1(y,z) = -\frac{1}{\pi} v_{11}^{'}(y) \sin \pi z, \tag{35}$$

$$p_1(y,z) = p_{11}(y) \cos \pi z. \tag{36}$$

In Eq. (35) the dash "$'$" denotes differentiation with respect to "y". We note that the velocity components (34) – (35) identically satisfy the continuity equation (29).

Substituting Eqs. (34) and (36) in Eqs. (31) and (32), we have

$$K\alpha R_e(v_{11}^{'''} - \pi^2 v_{11}^{'}) - v_{11}^{''} + \pi^2 v_{11} - \alpha R_e v_{11}^{'} = -R_e p_{11}^{'}, \tag{37}$$

$$K\alpha R_e(v_{11}^{''''} - \pi^2 v_{11}^{''}) - v_{11}^{'''} + \pi^2 v_{11}^{'} - \alpha R_e v_{11}^{''} = R_e \pi^2 p_{11}, \tag{38}$$

and the boundary conditions are

$$v_{11}(0) = -\alpha,\ v_{11}^{'}(0) = 0. \tag{39}$$

On eliminating the pressure p_{11} from Eqs. (37) and (38) we get the following differential equation:

$$K\alpha R_e\left(v_{11}^{'''''} - 2\pi^2 v_{11}^{'''} + \pi^4 v_{11}^{'}\right) - v_{11}^{''''} - \alpha R_e v_{11}^{'''} + 2\pi^2 v_{11}^{''} + \pi^2 \alpha R_e v_{11}^{'} - \pi^4 v_{11} = 0. \tag{40}$$

We assume

$$v_{11} = v_{110} + K v_{111} + O(K^2), \tag{41}$$

using Eq. (41) in Eq. (40) and solving resulting equation, we get the following solutions:

$$v_{110} = \frac{-\alpha}{(\pi-\lambda)}(\pi e^{-\lambda y} - \lambda e^{-\pi y}), \tag{42}$$

$$v_{111} = \frac{-\pi\lambda\alpha^2 R_e(\lambda+\pi)}{(\alpha R_e - 2\lambda)(\lambda-\pi)}(e^{-\lambda y} - e^{-\pi y} - (\pi-\lambda) y e^{-\lambda y}), \tag{43}$$

where

$$\lambda = \frac{\alpha R_e}{2} + \sqrt{\left(\frac{\alpha R_e}{2}\right)^2 + \pi^2}.$$

Substitution of Eqs. (42) and (43) in Eq. (41), yields

$$v_{11} = \frac{-\alpha}{(\pi-\lambda)}(\pi e^{-\lambda y} - \lambda e^{-\pi y}) - K\frac{\pi\lambda\alpha^2 R_e(\lambda+\pi)}{(\alpha R_e - 2\lambda)(\lambda-\pi)}(e^{-\lambda y} - e^{-\pi y} - (\pi-\lambda) y e^{-\lambda y}). \tag{44}$$

Similarly from Eqs. (44) and (38), we get

$$
\begin{aligned}
p_{11} &= \frac{-\alpha\lambda}{\pi(\lambda-\pi)}\left[\alpha\pi + K\frac{\alpha^2(\lambda+\pi)}{(2\lambda-\alpha R_e)}\left(K\alpha R_e\pi^4 - K\alpha R_e\pi^2 - 2\pi^3 - \alpha R_e\pi^2\right)\right]e^{-\pi y} \\
&\quad -\frac{\alpha\lambda}{\pi(\lambda-\pi)}\left[\begin{array}{c} K\alpha\lambda^3 - K\alpha\lambda\pi^2 \\ +K\frac{\alpha^2(\lambda+\pi)}{(2\lambda-\alpha R_e)}\left(\begin{array}{c} 3K\alpha R_e\lambda^4 - 4K\alpha R_e\lambda^3\pi - K\alpha R_e\lambda^2 + 2K\alpha R_e\lambda\pi \\ +2\lambda^3 - 3\lambda^2\pi - \alpha R_e\lambda^2 - 2\alpha R_e\lambda\pi + \pi^3 \end{array}\right)\end{array}\right]e^{-\lambda y} \\
&\quad +K\frac{\alpha^2(\lambda+\pi)}{(2\lambda-\alpha R_e)}\left[\begin{array}{c} -K\alpha R_e\lambda^4(\lambda-\pi) + K\alpha R_e\lambda^2\pi^2(\lambda-\pi) \\ -\lambda^3(\lambda-\pi) + \alpha R_e\lambda^2(\lambda-\pi) + \lambda\pi^2(\lambda-\pi)\end{array}\right]ye^{-\lambda y} \qquad (45)
\end{aligned}
$$

Substituting Eqs. (44) and (45) in Eqs. (34)-(36), we get

$$
v_1(y,z) = \frac{-\alpha}{(\pi-\lambda)}\left[\begin{array}{c} (\pi e^{-\lambda y} - \lambda e^{-\pi y}) + K\frac{\pi\lambda\alpha R_e(\lambda+\pi)}{(2\lambda-\alpha R_e)}(e^{-\lambda y} - e^{-\pi y} \\ -(\pi-\lambda)ye^{-\lambda y}) \end{array}\right]\cos\pi z, \qquad (46)
$$

$$
w_1(y,z) = \frac{-\alpha\lambda}{(\pi-\lambda)}\left[\begin{array}{c} (e^{-\lambda y} - e^{-\pi y}) + K\frac{\alpha R_e(\lambda+\pi)}{(2\lambda-\alpha R_e)}(\pi(e^{-\lambda y} - e^{-\pi y}) \\ -\lambda(\pi-\lambda)ye^{-\lambda y}) \end{array}\right]\sin\pi z, \qquad (47)
$$

$$
\begin{aligned}
p_1(y,z) &= \frac{-\alpha\lambda}{\pi(\lambda-\pi)}\left[\alpha\pi + K\frac{\alpha^2(\lambda+\pi)}{(2\lambda-R_e)}\left(K\alpha R_e\pi^4 - K\alpha R_e\pi^2 - 2\pi^3 - \alpha R_e\pi^2\right)\right]e^{-\pi y}\cos\pi z \\
&\quad -\frac{\alpha\lambda}{\pi(\lambda-\pi)}\left[\begin{array}{c} K\alpha\lambda^3 - K\alpha\lambda\pi^2 \\ +K\frac{\alpha^2(\lambda+\pi)}{(2\lambda-\alpha R_e)}\left(\begin{array}{c} 3K\alpha R_e\lambda^4 - 4K\alpha R_e\lambda^3\pi \\ -K\alpha R_e\lambda^2 + 2K\alpha R_e\lambda\pi + 2\lambda^3 \\ -3\lambda^2\pi - \alpha R_e\lambda^2 - 2\alpha R_e\lambda\pi + \pi^3 \end{array}\right)\end{array}\right]e^{-\lambda y}\cos\pi z \\
&\quad +K\frac{\alpha^2(\lambda+\pi)}{(2\lambda-\alpha R_e)}\left[\begin{array}{c} -K\alpha R_e\lambda^4(\lambda-\pi) \\ +K\alpha R_e\lambda^2\pi^2(\lambda-\pi) - \lambda^3(\lambda-\pi) \\ +\alpha R_e\lambda^2(\lambda-\pi) + \lambda\pi^2(\lambda-\pi)\end{array}\right]ye^{-\lambda y}\cos\pi z. \qquad (48)
\end{aligned}
$$

The Eqs. (46) and (47) present the cross-flow velocity distribution and pressure in Eq. (48) provide the input for the solution to the axial velocity. The viscous results [2] are recovered when $K \to 0$.

4.2 Main flow solution

The solution for the Eq. (30) can be expressed as

$$
u_1(y,z) = u_{11}(y)\cos\pi z. \qquad (49)
$$

The corresponding boundary conditions (33) are reduced to

$$\begin{aligned} u_{11} &= 0 \ at \ y=0, \\ u_{11} &= 0 \ as \ y \to \infty. \end{aligned} \tag{50}$$

Further we assume that

$$u_{11} = u_{110} + K u_{111} + O(K^2). \tag{51}$$

Then the boundary conditions (50) yield

$$\begin{aligned} u_{111} &= u_{110} = 0 \text{ at } y=0, \\ u_{111} &= u_{110} = 0 \text{ as } y \to \infty. \end{aligned} \tag{52}$$

Using Eqs. (25) , (46) and Eqs. (49)-(52) in Eq. (30), we get

$$\begin{aligned} u_1(y,z) &= \frac{\alpha R_e}{(\pi-\lambda)}\left[A_1 e^{-\lambda y} - A_2 e^{-(\lambda+\alpha R_e)y} + A_3 e^{-(\pi+\alpha R_e)y}\right]\cos\pi z \\ &+ \frac{K(\alpha R_e)^2}{(\pi-\lambda)}\left[\left(\frac{E_2}{2\lambda\alpha R_e} + \frac{E_3}{\pi\alpha R_e} + E_4 + E_6\right)e^{-\lambda y} - \left(\frac{E_1}{\alpha R_e - 2\lambda}\right) y e^{-\lambda y}\right]\cos\pi z \\ &- \frac{K(\alpha R_e)^2}{(\pi-\lambda)}\left[\begin{array}{c} \left(\frac{E_2}{2\lambda\alpha R_e} + E_4\right)e^{-(\lambda+\alpha R_e)y} + \left(\frac{E_3}{\pi\alpha R_e} + E_6\right)e^{-(\pi+\alpha R_e)y} \\ + E_5 y e^{-(\lambda+\alpha R_e)y} + E_7 y e^{-(\pi+\alpha R_e)y} \end{array}\right]\cos\pi z, \end{aligned} \tag{53}$$

where

$$A_1 = \left(\frac{\pi}{2\lambda} - \frac{\lambda}{\pi}\right),\ A_2 = \frac{\pi}{2\lambda},\ A_3 = \frac{\lambda}{\pi},\ E_1 = \left(\lambda^3 - \lambda\pi^2\right)\left(\frac{\pi}{2\lambda} - \frac{\lambda}{\pi}\right),$$

$$E_2 = -2\alpha^2 R_e^2 \pi + \frac{\pi(\lambda+\pi)}{2\lambda-\alpha R_e} - \frac{\pi(\lambda+\alpha R_e)^3}{2\lambda} + \frac{\pi^3(\lambda+\alpha R_e)}{2\lambda},$$

$$E_3 = 2(\alpha R_e)^2\lambda - \frac{\pi(\lambda+\pi)}{2\lambda-\alpha R_e} + \frac{\lambda(\pi+\alpha R_e)^3}{\pi} - \lambda\pi(\pi+\alpha R_e),$$

$$E_4 = \frac{2\lambda+\alpha R_e}{4\lambda^2(\alpha R_e)^2}\left(\frac{\pi(\lambda\alpha R_e)^2}{2\lambda-\alpha R_e} + \pi(\alpha R_e)^3\right),\ E_5 = \frac{1}{2\lambda\alpha R_e}\left(\frac{\pi(\lambda\alpha R_e)^2}{2\lambda-\alpha R_e} + \pi(\alpha R_e)^3\right),$$

$$E_6 = \frac{2\pi+\alpha R_e}{\pi^2(\alpha R_e)^2}\left(-\lambda(\alpha R_e)^3\right),\ E_7 = \frac{1}{\pi\alpha R_e}\left(-\lambda(\alpha R_e)^3\right).$$

Substituting Eqs. (25) and (53) in Eq. (26) , we get

$$
\begin{aligned}
u(y,z) &= 1-e^{-\alpha R_e y}-K\left(\alpha R_e\right)^3 y e^{-\alpha R_e y}+\varepsilon\frac{\alpha R_e}{(\pi-\lambda)}\left[A_1 e^{-\lambda y}-A_2 e^{-(\lambda+\alpha R_e)y}+A_3 e^{-(\pi+\alpha R_e)y}\right]\cos\pi z \\
&+\varepsilon\frac{K\left(\alpha R_e\right)^2}{(\pi-\lambda)}\left[\left(\frac{E_2}{2\lambda\alpha R_e}+\frac{E_3}{\pi\alpha R_e}+E_4+E_6\right)e^{-\lambda y}-\left(\frac{E_1}{\alpha R_e-2\lambda}\right)ye^{-\lambda y}\right]\cos\pi z \\
&-\varepsilon\frac{K\left(\alpha R_e\right)^2}{(\pi-\lambda)}\left[\begin{array}{c}\left(\frac{E_2}{2\lambda\alpha R_e}+E_4\right)e^{-(\lambda+\alpha R_e)y}+\left(\frac{E_3}{\pi\alpha R_e}+E_6\right)e^{-(\pi+\alpha R_e)y} \\ +E_5 y e^{-(\lambda+\alpha R_e)y}+E_7 y e^{-(\pi+\alpha R_e)y}\end{array}\right]\cos\pi z, \qquad (54)
\end{aligned}
$$

It should be noted that the limiting velocity u_1 as $K\to 0$, differs from that computed by Gersten and Gross [2]. This is due to some calculation mistake in their work.

4.3 Shear stress components

The expressions for the shear stress components in the x-direction and z-direction can be expressed as follows:

$$
\begin{aligned}
C_{fx} &= \frac{(\frac{\partial u}{\partial y})_{y=0}}{\alpha R_e} \\
&= F_0+\epsilon\cos\pi z-\epsilon F_1(R_e)\cos\pi z, \qquad (55)
\end{aligned}
$$

and

$$
\begin{aligned}
C_{fz} &= \frac{\mu(\frac{\partial w}{\partial y})_{y=0}}{\alpha} \\
&= -\epsilon F_2(R_e)\sin\pi z. \qquad (56)
\end{aligned}
$$

The functions $F_1(R_e)$ and $F_2(R_e)$ are given by

$$
\begin{aligned}
F_1(R_e) &= \frac{(\lambda+\pi)\pi}{2\lambda^2}-\frac{KR}{(\pi-\lambda)}\left[-\lambda\left(\frac{E_2}{2\lambda\alpha R_e}+\frac{E_3}{\pi\alpha R_e}+E_4+E_6\right)-\frac{E_1}{\alpha R_e-2\lambda}\right. \\
&\qquad +\frac{(\lambda+\alpha R_e)E_2}{2\lambda\alpha R_e}+\frac{(\pi+\alpha R_e)E_3}{\pi\alpha R_e} \\
&\qquad \left.+(\lambda+\alpha R_e)E_4-E_5+(\pi+\alpha R_e)E_6-E_7\right]. \qquad (57)
\end{aligned}
$$

It is worth mentioning that the skin friction factor $F_1\left(R_e\right)$ when $K\to 0$ reduces to steady state value of [7]. It is also indicated that limiting result as $K\to 0$ differs from that found by Gersten and Gross [2]. This happens due to some calculation mistake in their work.

$$
F_2(R_e)=\lambda\left(1+K\frac{\lambda\left(\alpha R_e\right)^2}{\alpha R_e-2\lambda}\right). \qquad (58)
$$

The limiting result of $F_2(R_e)$ as $K \to 0$ is identical to that obtained by Gersten and Gross [2] and steady state value presented by Singh et al. [7].

5 Results and discussion

The effects of dimensionless parameters such as elastic parameter K, Reynolds number R_e and suction parameter α on velocity component u are shown in Figs. $(2)-(4)$. Skin friction factors $F_1(R_e)$, $F_2(R_e)$ are presented graphically in Figs. $(5)-(8)$. Fig. (2) shows that the velocity component u decreases with the increase of dimensionless parameter K which was expected naturally. For a particular value of K, the velocity component u increases gradually to attain maximum value equal to unity. The Fig. (3) shows the effect of Reynolds number R_e on the main flow velocity component u. It is observed from this figure that velocity is increasing function of R_e. However, velocity decreases in the vicinity of the plate. Moreover, backflow is observed for $R_e > 30$. The influence of suction parameter α on the velocity component u is demonstrated in Fig. (4). The main flow velocity component u increases as the suction parameter α increases which was expected naturally. However, it decreases near the plate and then increases exponentially. Backflow near the plate is observed for $\alpha > 0.3$. Furthermore, $u \to 1$ as $y \to \infty$.

The effect of dimensionless parameters K and α on the shear stress component $F_1(R_e)$ are depicted in Figs. (5) and (6) respectively. The Fig. (5) shows that the shear stress component $F_1(R_e)$ increases with an increase in K. It decreases as R_e increases from zero to some value(depending upon K) of R_e, then increases exponentially and tends to infinity. Similar effect of α on F_1 is noted in Fig. (6). Of course, F_1 tends to be linearized as $\alpha \to 0.1$. Moreover, $F_1(R_e) \to 1$ as $R_e \to 0$.

The $F_2(R_e)$ and its asymptotic limits are shown in Figs. (7) and (8). In Fig. (7) the dimensionless parameter α is fixed and K is varied. In Fig. (8) the role of these dimensionless parameters is interchanged. Fig. (7) shows great influence of elastic parameter on $F_2(R_e)$ which is decreasing function of elastic parameter K. Moreover, F_2 increases as R_e increases from zero to some value(depending upon K) of R_e, then decreases for higher values. It is shown in Fig. (8) that $F_2(R_e)$ initially increases and then decreases for any fix value of α. Also, it can be perceived that F_2 tends to linearized as $\alpha \to 0.1$. The Fig. (9) demonstrates that the transverse wall shear stress, which results from the secondary flow normal to the main flow direction, disappears due to symmetry at the points of maximum and minimum suction velocity. The effect of elastic parameter K and suction parameter α on the velocity component w_1 are tabulated in Table 1. It is observed that w_1 increases as α increases. However an opposite effect of K on w_1 is noted. It also decreases in the y-direction.

6 Conclusion

The three-dimensional incompressible laminar flow of a second grade fluid past a wall is analyzed. A suction with a slightly sinusoidal transverse suction velocity distribution at the wall is employed. Approximate solutions for main flow, cross flow and pressure are presented. For the asymptotic flow condition far downstream the components of the wall shear stress are computed. The major findings of the present study are as follow:

- When K increases the main flow velocity u decreases. In the limiting case, when $y \to \infty$, it(main flow velocity) approaches to unity
- When R_e increases the main flow velocity u also increases
- Shear stress components tend to be linearized as $\alpha \to 0.1$
- The shear stress components in the direction of main flow $F_1(R_e)$ and the function $F_2(R_e)$ which characterizes the wall shear stress in the z-direction, strongly depend upon both elastic parameter K and suction parameter α
- Reynolds number R_e and suction parameter α provide a mechanism to control the backflow
- When $K \to 0$, the viscous results for cross flow [2] are recovered
- The limiting main flow velocity u when $K \to 0$ differs from that obtained by Gersten and Gross [2] due to some calculation mistake in their work
- The steady state value of skin friction factor in main flow direction [7] is recovered when $K \to 0$. It, however, differs from that obtained by Gersten and Gross due to some computational mistake in their work
- The limiting result of $F_2(R_e)$ as $K \to 0$ is identical to that obtained by Gersten and Gross [2] and steady state value presented by Singh et al. [7]

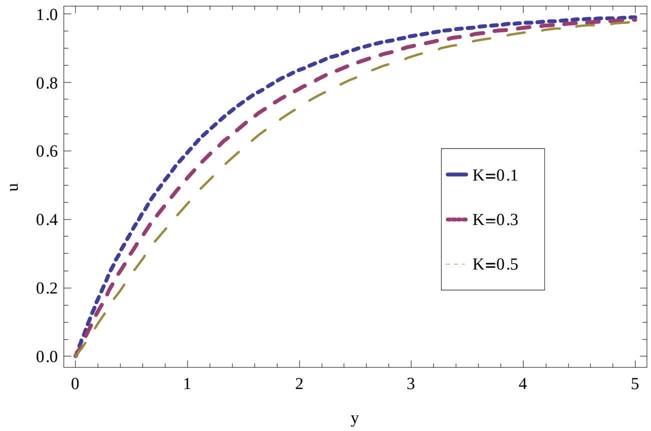


Fig. (2) Variation of u at $\alpha = 0.1,\ R_e = 10,\ \epsilon = 0.1$ and $z = 0$ for different values of K

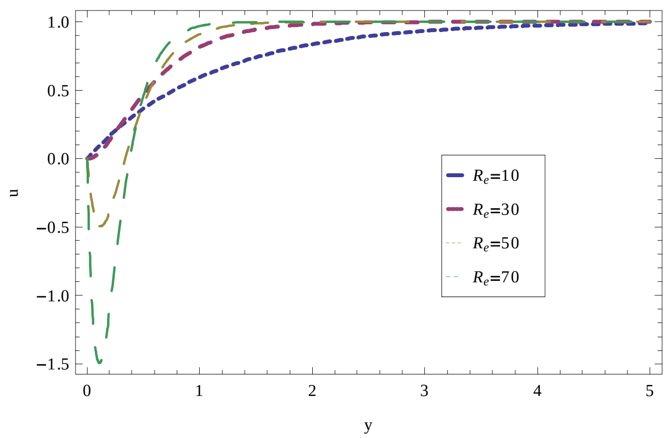


Fig. (3) Variation of u at $\alpha = 0.1,\ K = 0.1,\ \epsilon = 0.1$ and $z = 0$ for different values of R_e

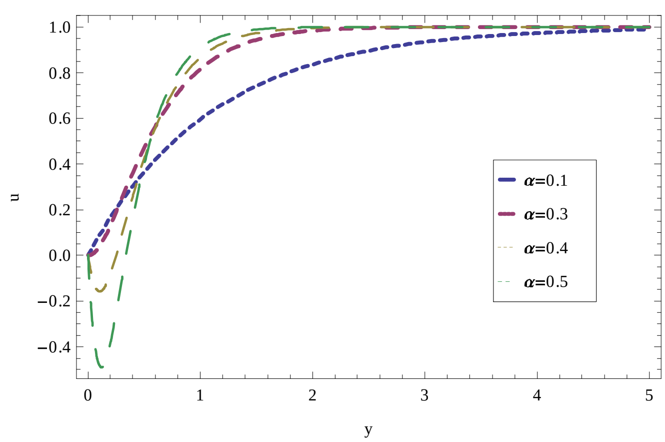


Fig. (4) Variation of u at $K = 0.1,\ R_e = 10,\ \epsilon = 0.1$ and $z = 0$ for different values of α

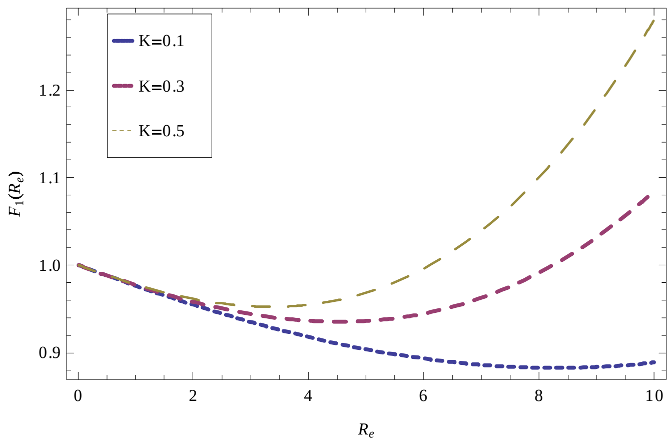


Fig. (5) Variation of $F_1(R_e)$ at $\alpha = 0.1$ for different values of K

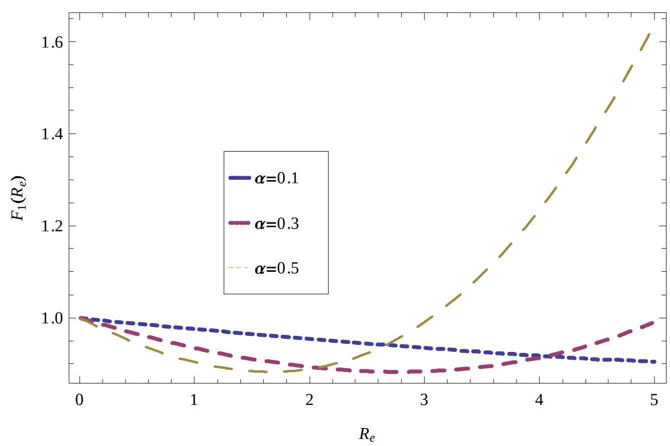


Fig. (6) Variation of $F_1(R_e)$ at $K = 0.1$ for different values of α

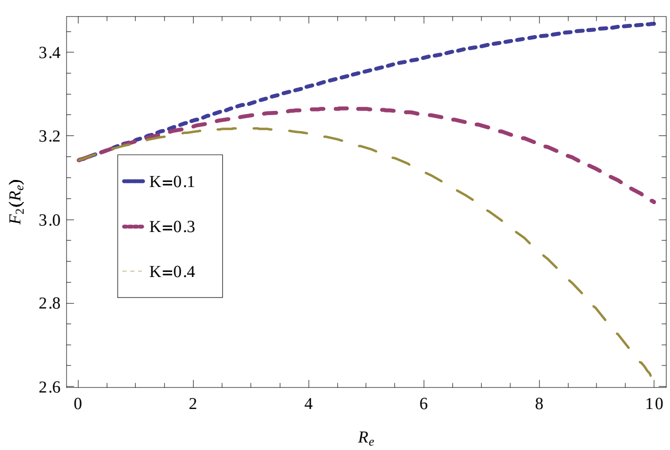


Fig. (7) Variation of $F_2(R_e)$ at $\alpha = 0.1$ for different values of K

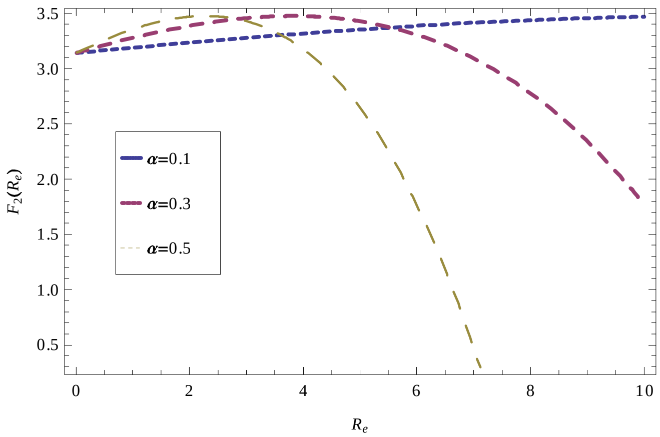
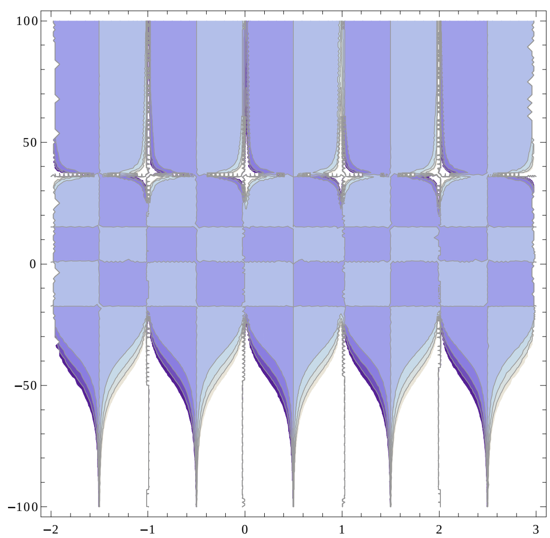


Fig. (8) Variation of $F_2(R_e)$ at $K = 0.1$ for different values of α



$Fig.$(9) Flow streamlines on the surface of the flat plate for $K = 0.1$ and $\alpha = 0.1$

Table 1: Effects of K and α on transverse velocity component w for $\epsilon = 0.1$, $z = -0.5$ and R_e =10.

y	K=0.1, α=0.1	K=0.1,α=0.3	K=0.1,α=0.5	K=0.5,α=0.1	K=0.5,α=0.3	K=0.5,α=0.5
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.4	0.00765642	0.0229064	0.0379688	0.00720078	0.0186956	0.0259742
0.8	0.00425012	0.0120589	0.0188772	0.00399335	0.00973349	0.0123868
1.2	0.00176929	0.00475647	0.00701495	0.00166076	0.00379298	0.00437853
1.6	0.000654638	0.00166584	0.00230802	0.000613862	0.00131089	0.00135524
2.0	0.000227056	0.000546312	0.000708632	0.000212693	0.000423678	0.000385539

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