

Numerical Solution of Volterra Integral Equation and its Error Estimates Via Spectral Method

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Abstract

In this article, numerical solution of Volterra integral equations is considered. A new approach in the application of spectral method is proposed, wherein Chebyshev polynomial of the first kind $T_k(x)$ serves as the basis function. Essentially, the method is based on the approach of series solution where coefficients of $T_k(x)$ in the residual equations are correspondingly equated to yield system of equations. Expression for error estimates which effectively serves as upper bound for accrued errors is arrived at. To illustrate the accuracy and effectiveness of the method and its error estimates, numerical examples on some standard integral equations are given.

Keywords: Spectral method; Chebyshev basis function; Coefficients; Volterra Integral equations; Error estimates.

1. Introduction

On a general note, the term integral equation refers to an equation where an unknown function occurs under an integral.

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The standard type of such equation in $u(x)$ is of the form:

$$\alpha(x)u(x) + \lambda \int_a^b k(x, t) u(t) dt = f(x) \tag{1}$$

Where a and b are limits of integration, λ is a constant parameter. $k(x, t)$ is a known function of 2 variables called the kernel or the nucleus of the integral equation. It is defined in the square: $\Pi = \{(x, t) : a \leq x \leq b, a \leq t \leq b\}$

$f(x)$ is a given function that corresponds to an external force acting on the system. If $f(x)$ is identically zero, the resulting equation is called homogeneous.

Solving equation (1) amounts to determining a function $u(x)$ such that (1) is satisfied for all points within the interval; $a \leq x \leq b$. (Herman Brunner [1]).

As established by Grewal [2], as it is for differential equations, there are several integral equations for which analytic solution is not feasible, even in the availability of such solutions, the computation cost may be so enormous that numerical approach becomes the best and most viable alternative. The aim of this work is however to introduce a new technique in the application of spectral methods such that the method will yield approximate solution that is efficient, effective and with minimal computational cost.

Spectral methods are based on the representation of a real, continuous function $f(x)$ on some interval as an expansion in an orthogonal set of functions $\phi_n(x)$ i. e

$$f(x) = \sum_{k=0}^{\infty} c_k \phi_k(x), \quad x \in [a, b] \tag{2}$$

Where the polynomials $\phi(x)$ are orthogonal i. e.

$$\int_a^b w(x) \phi_n(x) \phi_m(x) dx = \delta_{nm} \tag{3}$$

With $w(x)$ as the weight function and the kronecker delta is defined by:

$$\delta_{nm} = \begin{cases} 1, & m=n \\ 0, & m \neq n \end{cases} \tag{4}$$

As illustrated in [3], the function of interest $u(x)$ is approximated with the finite sum

$$u_n(x) = \sum_{k=0}^n c_k T_k(x), \quad x \in [a, b] \quad (5)$$

Where $T_k(x)$ is the Chebyshev polynomials of the first kind and c_k are coefficients of expansion. This is directly obtained from Fourier cosine series in Chebyshev polynomial form. (see ref [2][3][4][5]). The expansion coefficients c_k are occasionally referred to as the generalized Fourier coefficients. As exemplified in [3], one clear advantage that spectral methods have over finite difference methods is that once approximate spectral coefficients have been found, the approximate solution can immediately be evaluated at any point in the range of integration, whereas to evaluate a finite-difference solution at an intermediate point requires a further step of interpolation. In addition, it has been established over the years that these methods possess ability to tackle a wide variety of problems. In particular, they yield accurate results with only moderate computational resources.

2. Classification of Integral Equations

Integral equations appear in many varieties, the types depend mainly on the limits of integration, the kernel of the equation and the appearance of unknown function $u(x)$. If the limits of integration are fixed, the integral equation is called a Fredholm integral equation, a typical example is given in equation (1), where a and b are constants. But if at least one limit is a variable, the equation is called a Volterra integral equation given in the form:

$$\alpha(x)u(x) + \lambda \int_a^x k(x, t)u(t) dt = f(x) \quad (6)$$

In addition to this, integral equations can further be classified into 2 kinds taking the following generic forms:

$$\int_a^b k(x, t)u(t) dt = f(x) \quad (7)$$

That is $\alpha \equiv 0$ in equation (1) such that u appears only under the integral sign the integral equation is called a Fredholm integral equation of the first kind.

However, for Fredholm integral equations of the second kind, $\alpha \neq 0$ hence the unknown function u appears both inside and outside the integral sign. This therefore takes the form:

$$\alpha(x)u(x) + \lambda \int_a^b k(x, t)u(t) dt = f(x) \quad (8)$$

3. Evaluation Of Chebyshev Terms

Throughout the course of this work, we constantly need to express product like $f(x)T_k(x)$ and the integral of $T_k(x)$. For the evaluation of such terms it has been observed that for an efficient and stable execution, the secret is to avoid rewriting Chebyshev polynomials in terms of powers of x and to operate wherever possible with the Chebyshev polynomials themselves (Clenshaw [6]). As a result of this, this work is based on expressing all terms in the equation in series of Chebyshev polynomials.

For the integral of $T_k(x)$, to allow for effective evaluation, we express this as follows:

$$\begin{aligned} \int T_k(x) dx &= \int -\cos k \theta \sin \theta d\theta \\ &= -\frac{1}{2} \int [\sin(k+1)\theta - \sin(k-1)\theta] d\theta \\ &= \frac{1}{2} \left[\frac{\cos(k+1)\theta}{k+1} - \frac{\cos|k-1|\theta}{k-1} \right] \end{aligned}$$

Hence,

$$\int T_k(x) dx = \begin{cases} \frac{1}{2} \left[\frac{T_{k+1}(x)}{k+1} - \frac{T_{|k-1|}(x)}{k-1} \right]; & k \neq 1 \\ \frac{1}{4} T_2(x); & k = 1 \end{cases} \quad (9)$$

Also it is frequently necessary to be able to multiply Chebyshev polynomials by factors such as x , $1-x$ and $1-x^2$ and to express the result in terms of Chebyshev polynomials.

$$\begin{aligned} xT_k(x) &= \cos \theta \cos \theta = \frac{1}{2} [\cos(k+1)\theta + \cos|k-1|\theta] \\ &= \frac{1}{2} (T_{k+1}(x) + T_{|k-1|}(x)) \end{aligned} \quad (10)$$

On a general note, expression for $x^m T_k(x)$, as illustrated in [3], is given by:

$$x^m T_k(x) = 2^{-m} \sum_{r=0}^m \binom{m}{r} T_{k-m-2r}(x) \quad (m < k) \quad (11)$$

In circumstances where the variable coefficient is a non-polynomial term, we adopt a technique in [4] for the expression of $f(x)T_k(x)$ where $f(x)$ is first written in Taylor series form.

Also, the power x^n that makes up $f(x)$ on the LHS of equation (1) is expressed in terms of the Chebyshev polynomials of degrees up to n . As established by Mason and Handscomb [3], this is given by:

$$x^n = 2^{1-n} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} T_{n-2k}(x) \tag{12}$$

4. Spectral Methods

Spectral methods are part of several methods founded on orthogonal expansions, in this study we consider the expansions in Chebyshev polynomials of the first kind. In this method, the idea is to write the solution to the integral (1) as a series of ‘‘Chebyshev basis functions’’ i. e.

$$\hat{u}_n(x) = \frac{1}{2}c_0 + \sum_{k=1}^n c_k T_k(x) \tag{13}$$

Equation (13) is referred to as the trial solution while the expansion coefficients $\{c_k\}$ is referred to as the spectral space representation of the function.

It is to be noted that the implementation of spectral methods is normally accomplished with either Collocation, Galerkin or a Tau approach. (See ref [1][5][9][10]). But in this study, a new approach of factorizing the coefficients is applied as an implementation technique.

Equation (13) is substituted into (1) to yield:

$$\alpha(x)\hat{u}(x) + \lambda \int_a^b k(x,t)\hat{u}(t)dt = f(x) \tag{14}$$

which becomes

$$\alpha(x) \left[\frac{1}{2}c_0 + \sum_{k=1}^n c_k T_k(x) \right] + \lambda \int_a^b k(x,t) \left[\frac{1}{2}c_0 + \sum_{k=1}^n c_k T_k(x) \right] dt \approx f(x) \tag{15}$$

With the aid of approaches in ([4][6][8]) products like $\alpha(x)T_k(x)$ and integral $\int k(x,t)T_k(x)$ are easily resolved into series of Chebyshev polynomials. Also $f(x)$ whether as polynomials, trigonometry, exponential and logarithmic function is easily resolved into series of Chebyshev polynomials. Equation (15) therefore becomes an equation in series of Chebyshev polynomials $T_k(x)$ of varying degree on both sides. Coefficients of each $T_k(x)$ are thereafter equated correspondingly to yield a system of equations:

$$A X = b \tag{16}$$

Where A is the matrix of coefficients, X is the column vector containing the expansion coefficients $\{c_n\}$ and b is the corresponding coefficients of $T_k(x)$ after $f(x)$ has been converted into series of Chebyshev polynomials.

Solving (16) using any algebraic solution methods yields numerical values for c_0, c_1, \dots, c_n . These are thereafter substituted into (5) to yield an approximate solution to integral equation (1).

5. Error Estimates

There are several studies concerning the convergence of spectral methods. These include (Cheney [7], and Wazwaz [9]). Discussion on the theory of approximation from an historical perspective can be found in [8]. The minimax property of $T_k(x)$ enhances the series in the trial solution to provide an accurate approximation to $u(x)$ with small number of terms.

However, it is to be noted that if the method is efficient, the absolute values of the coefficients $|c_n|$ decrease rapidly with increasing n , this is realised in this work and error estimate based on the sizes of Chebyshev coefficients is established.

It is vividly observed in the course of this work that the resulting errors from the solved problems yield error estimate that is governed by:

$$|E_n| \leq |c_0 c_n n^{-1}| \quad (17)$$

6. Numerical Experiment

The described method with our new approach are applied to six test problems listed below. This is basically to illustrate the efficiency of this technique and also to compute exact error which will serve to determine the validity of the formulation for error estimates.

Example 1.

$$2 + x - 2e^x + xe^x = \int_0^x (x-t)y(t) dt$$

The analytical solution is $y(x) = xe^x$

Example 2.

$$x - \frac{1}{2}x^2 - \ln(1+x) + x^2 \ln(1+x)^x = \int_0^x 2t y(t) dt$$

The analytical solution is $y(x) = \ln(1+x)$

Example 3.

$$y(x) = 2e^x - 2 - x + \int_0^x (x-t) y(t) dt$$

The analytical solution is $y(x) = xe^x$

Example 4.

$$\phi(x) - \int_0^x \frac{(1+x)}{1+t} \phi(t) dt = 1 - x - \frac{3}{2}x^2 + \frac{x^3}{2} \quad ; \quad 0 < x < 1$$

The analytical solution is $\phi(x) = 1 - x^2$

Example 5.

$$y(x) = e^{-x} - 2 \int_0^x \cos(x-t) y(t) dt$$

The exact solution is $y(x) = e^{-x} (1-x)^2$

Example 6.

$$y(x) = 4x - 3 \int_0^x y(t) \sin(x-t) dt$$

The exact solution is $y(x) = x + \frac{3}{2} \sin 2x$

7. Numerical Solutions to the Problems

In this section, maximum error with the corresponding error estimates obtained by the use of equation (17) are tabulated for each solved problem.

Table 7.1: Table of errors for example 1

n	Maximum error	Error estimates
4	9.1817e-004	1.1880e-003
6	1.2914e-006	2.4433e-006
8	5.0494e-010	1.6489e-009
10	4.0378e-014	2.5328e-012

Table 7.2: Table of errors for example 2

n	Maximum error	Error estimates
4	4.5330e-004	6.1294e-004
6	2.0426e-005	3.5974e-005
8	4.8223e-008	3.4838e-007
10	3.9632e-010	5.7352e-009

Table 7.3: Table of errors for example 3

N	Maximum error	Error estimates
4	3.0502e-004	1.2616e-003
6	6.6005e-007	2.5637e-006
8	5.7364e-010	2.8154e-009
10	5.0188e-013	1.1451e-012

Table 7.4:Table of errors for example 4

N	Maximum error	Error estimates
4	0	0
6	0	0
8	0	0
10	0	0

Table 7.5: Table of errors for example 5

N	Maximum error	Error estimates
4	5.4303e-004	1.2406e-003
6	4.2182e-006	8.1274e-005
8	2.8226e-009	3.2139e-008
10	3.6431e-011	4.1236e-010

Table 7.6: Table of errors for example 6

N	Maximum error	Error estimates
4	9.6371e-004	4.2969e-002
6	5.2162e-006	3.8471e-005
8	1.3432e-008	3.9476e-007
10	1.4832e-010	6.4312e-009

8. Conclusion

The Spectral method with a new implementation technique has been proposed and applied to integral equations. Formulation for error estimates with the use of spectral coefficients has also been established. A look at table 7.1 – 7.6 shows that the technique performs effectively well in the solution of these problems as the produced errors are very minimal. In addition to this, a consideration of this technique depicts the simplicity in its application and a very low computational cost.

The above tables equally depicts that the formulation for error estimates is very efficient in application as it provides a reliable bound for the computed error.

Also, the assertion of [9] about the convergence of spectral methods is equally observed in this technique as values for C_k progressively and rapidly drops as k increases.

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