American Academic Scientific Research Journal for Engineering, Technology, and Sciences
ISSN (Print) 2313-4410, ISSN (Online) 2313-4402

https://asrjetsjournal.org/index.php/American_Scientific_Journal/index

Solution of the Generalized Pythagorean Equation; $x^2 + y^2 = n!$ and Formula π by Number Theory

Deval Sidi Mohamed*

Department of Mathematics, Faculty of Science and Technology, University of Nouakchott

Email:sidideval@gmail.com

Abstract

In this paper, we are interested in studying the generalized Pythagorean equation:

$$x^2 + y^2 = n! \qquad n \in \mathbb{N} \qquad (\mathrm{x,y}) \in \mathrm{Z}^2.$$

Furthermore, the non-Principe Dirichlet character module 4, defined as follows:

$$\chi(\mathrm{d}) = \begin{cases} 1 & if \;\; d \equiv 1(mod4) \\ -1 & if \;\; d \equiv 3(mod4) \\ 0 & if \quad d \; is \; even \end{cases}$$

Using Jacobi's Two-Square Theorem, we show that the total number of lattices in the circle of radius $r = \sqrt{N}$ is equal;

$$\sum_{n=1}^{N} \sum_{d|n} \chi(d).$$

Finally, Application: the formula for π by number theory is given by

$$\pi = \lim_{N \to +\infty} \frac{4}{N} \left(\sum_{n=1}^{N} \chi(\mathrm{d}) \right) = 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots).$$

Keywords: Fermat's theorem; Pythagorean equation; Jacobi's Two-Square Theorem; Formula π by number theory.

MSC 2020: Primary: 11A05;11A07, Secondary:11A41 .

Received: 4/14/2026
Accepted: 6/14/2026
Published: 6/24/2026

---

* *Corresponding author.*

1.Introduction

Leonhard Euler announced his successful proof of Fermat’s theorem on sums of two squares, which states that an odd prime p can be expressed as x^2+y^2 if and only if $p \equiv 1(\bmod 4)$ in 1749, Euler, in his proof, based on the method of infinite descent, see [2,3,4].

Furthermore, for more details on Fermat's theorem on sums of two squares, see [5,6,9]. It is well-known that the general integer solution to the Pythagorean equation

$$x^2+y^2=z^2$$

Is, allowing an interchange of x and y:

$$x=m^2-n^2, \quad y=2nm, \quad z=m^2+n^2.$$

Properties of such triples, such as their connections to matrix generators and Fibonacci numbers, add to this subjects mystique; see [1,9]. In this paper, let $a,\ b$ be two integers such that:

$$a \geq 0, \quad a<b, \quad d \geq 2 \quad and \quad GCD(a,\mathrm{b})=1.$$

p is a prime number. We then introduce the following notations $P_n(a,d)$, and $q_n(a,d)$, which define by:

$$P_n(a,b)=\frac{\prod_{\frac{p}{\mathrm{d}}} p^{E\left(\frac{n}{p-1}\right)} \prod_{k=1}^{n}(a+kd)}{n!} \tag{1.1}$$

And

$$q_n(a,d)=\frac{\prod_{\frac{p}{d}} p^{E\left(\frac{\mathrm{n}-1}{\mathrm{P}-1}\right)} \prod_{k=1}^{n}(a+(k-1)d)}{n!} \tag{1.2}$$

First, we show that $P_n(a,d)$ (given the condition $GCD(a,b)=1$) is an integer. Let p be an arbitrary prime number; We assume that the numerator and denominator of (1.1) be exactly divisible by the $U_P(n)$-th and $V_p(n)$ -th powers of p , respectively; it is to be shown that $U_P(n) \geq V_P(n)$. If p/d we have:

$$V_P(n)=\sum_{i=1}^{+\infty} E\left(\frac{n}{p^i}\right) \leq \sum_{i=1}^{+\infty} \frac{n}{p^i}=\frac{n}{p-1}$$

Then $V_P(n) \leq E(\frac{\mathrm{n}}{P-1})$ Which proves our claim for p/d.

However, if $p \nmid d$, then $U_P(n)$ is clearly equal to the number of factors of the second product divisible by p in the numerator of (1.1), plus the number of factors of the same product divisible by p^2 plus etc. Now, the congruence:

$$a + kd \equiv\ 0\ mod(p^r) \qquad (1.3)$$

admits at least $E(\frac{n}{p^r})$ solutions for k in the considered interval $1 \leq k \leq n$, since this interval contains many complete systems of residues modulo p^r; therefore, in this case:

$$U_P(n) \geq \sum_{i=1}^{+\infty} E\,(\frac{n}{p^i}) = V_P(n)$$

We conclude that $P_n(a,b)$ is an integer number. Remarque: $E(x) = \lfloor x \rfloor$, the entire part of x.

2.The expression of $T_n(a,b)$

Let, $p_1, p_2, ..., p_h$,prime numbers $< d$ that p_i do not divide d.

The numbers, q_1 , q , ..., q_h are given by the following condition: $p_l q_l \equiv a\ mod\ (d)$, $q_l < d$, (l=1,2,....,h).

We consider the expression:

$$T_n(a,d) = \frac{\mathrm{P}_n(a,d)}{P_{E(\frac{n}{p_1})}(q_1,d)P_{E(\frac{n}{p_2})}(q_2,d)\ldots P_{E(\frac{n}{p_h})}(q_h,d)}$$

3.Some technical lemmas and Theorems

In this section, we will discuss some lemmas and theorems of technique.

Lemma 1: *We have:*

$$P_n(a,d) = \alpha^{\mathrm{n+o(n)}}$$

$Such\ that$: $\alpha = \alpha\ (d) = d \prod_{p/d} p^{\frac{1}{p-1}}$.

Proof:

We have:$n = E(\frac{n}{p-1})(p-1) + r,\ r \leq p-2,\ (if\ \ n\ > p-1)$;

$$\Rightarrow n - (p-2) \leq E(\frac{n}{p-1})(p-1)$$

$$\Rightarrow \frac{n}{p-1} - \frac{p-2}{p-1} \le E(\frac{n}{p-1}) \le \frac{n}{p-1}$$

$$if\ n \le p-1, \Rightarrow \frac{n}{p-1} - \frac{p-2}{p-1} \le E(\frac{n}{p-1}) \le \frac{n}{p-1}$$

On the other hand

$$kd \le a + kd \le (k+1)d, k \in \{1,2,\ldots.,n\}$$

Then: $\prod_{p/d} p^{-\frac{p-2}{p-1}} (d \prod_{p/d} p^{\frac{1}{p-1}})^n \le P_n(a,d) \le (n+1)(d \prod_{p/d} p^{\frac{1}{p-1}})^n$. (1.4)

We notice: $n + 1 = \alpha^{\frac{\ln(n+1)}{\ln\alpha}}$

Lemma 2:

Let $p^{W_p(n)}$ *be the highest power of* p *that divides the number* $P_n(a,d)$, *Then*

$$p^{W_p(n)} \le (n+1)d$$

Proof:

We have $W_p(n) = U_P(n) - V_P(n)$.

If $t_{p^r}(n)$ is the number of congruence solution (1.3) in the interval, $1 \le k \le n$.

And p does not-divide d then, $W_p(n) = \sum_{i=1}^{+\infty}\{ t_{p^i}(n) - E(\frac{n}{p^i})\}$.

Let $r_0 \in N^*$, such that: $p^{r_0} \le a + nd < p^{r_0+1}$;

$$\Rightarrow E(\frac{n}{p^r}) = t_{p^r}(n) = 0, \forall r > r_0.$$

If $r \le r_0$, put $k_0 = min\{k,\ a + kd \equiv\ 0\ mod(p^r)\}$;

$\Rightarrow p^r/(a + k_0 d + p^r d), p^r/(a + k_0 d + 2p^r d), \ldots\ldots, p^r/(a + k_0 d + E(\frac{n}{p^r})p^r d)$.

$$then, \qquad t_{p^r}(n) \le E(\frac{n}{p^r}) + 1.$$

$$\Rightarrow W_p(n) \le r_0.$$

If p/d

then,

$$\begin{aligned}
\mathrm{W_p}(n) &= E(\frac{n}{p-1}) - V_P(n) \\
&\leq \{\frac{n}{p} - E(\frac{n}{p})\} + \{\frac{n}{p^2} - E(\frac{\mathrm{n}}{p^2})\} + \ldots + \{\frac{n}{p^{r_0}} - E(\frac{n}{p^{\mathrm{r}_0}})\} \\
&\quad + \sum_{i=r_0+1}^{+\infty} \frac{n}{\mathrm{p^i}} \\
&\leq r_0 + \frac{n}{p^{r_0}} \sum_{i=1}^{+\infty} \frac{1}{p^i} = r_0 + \frac{n}{p^{r_0}} \cdot \frac{1}{p-1} \\
&< r_0 + \frac{n}{a+nd} \cdot \frac{p}{p-1} \leq r_0 + \frac{2}{d} \leq r_0 + 1.
\end{aligned}$$

So:

$$p^{\mathrm{W}_p(n)} \leq p^{r_0} \leq a + nd \leq (n+1)d$$

Lemma 3: *Let* $n \geq d,\ p > \sqrt{(n+1)d}$ *and*, $p \equiv \ \ a \ \ mod(d)$.

If p belongs to an interval:

$$\frac{a+nd}{1+kd} < p \leq \frac{n}{k} \qquad (1.5)$$

Then, $P_n(a,d)$ *is not divisible by p*

If p belongs to an interval:

$$\frac{n}{k+1} < p \leq \frac{a+(n+1)d}{1+kd} \qquad (1.6)$$

Then $P_n(a,d)$ *is divisible by* p *.*

Proof: We have $\mathrm{p}^2 > (n+1)d > \mathrm{n}$, then:

$$E(\frac{\mathrm{n}}{p^2}) = E(\frac{n}{p^3}) = \ldots = 0, \text{ and } t_{p^2}(n) = t_{p^3}(n) = \ldots = 0$$

On the other hand:

$$\mathrm{p} > \sqrt{\mathrm{nd}} \geq d \Rightarrow GCD(p,d) = 1.$$

$$\text{Then } p^{\mathrm{W}_p(n)} = t_p(n) - E(\frac{n}{p})$$

By (1.5) we have: $\mathrm{k} \leq \frac{n}{p} \Rightarrow k \leq E(\frac{\mathrm{n}}{p})$.

Another: $a + xd \equiv 0 \ mod(d)$ (i.e) $a + xd = py$;

$$p \equiv a \; mod(d) \equiv \; py.$$

Since, $GCD(p, d) = 1, \Rightarrow y \equiv 1 \;\; mod(d)$

By (1.5), we find; $\quad$ p $(1 + kd) > a + nd \Rightarrow y < 1 + kd$

$$\Rightarrow t_p(n) \leq k \text{ , then } \; p^{\mathrm{W}_p(n)} = 0$$

By (1.6) we have;

$$\frac{n}{p} < k + 1 \Rightarrow E(\frac{n}{p}) \leq k.$$

And $p(1 + kd) < a + (n + 1), p(1 + kd) \equiv a \; mod(d)$

Then, $\quad p(1 + kd) \leq a + nd$

On the other hand: $p > d > a,$ and $p \equiv a \; mod(d)$

$$\mathrm{p} \geq a + d.$$

Then, p , $p(1 + d)$, ..., $p(1 + kd)$ are numbers of the form $a + xd$ and each is divisible by p . $\Rightarrow \mathrm{t}_p(n) \geq k + 1.$

$\Rightarrow \mathrm{W}_p(n) \geq 1$ and On the other hand, $\mathrm{W}_p(n) \leq 1$

So: $\qquad \mathrm{W}_p(n) = 1$

Lemma 4: *We have:*

$$\mathrm{T}_n(a, d) = \alpha^{\,(1-\,\sigma)n + o(n)}.$$

$$or \;\; \sigma = \; \sigma(d) = \sum_{p \; does \; not \; divide \; p \quad p<d} \frac{1}{p}$$

Proof:

$$\begin{aligned} T_n(a, d) &= \alpha^{n - E(\frac{\mathrm{n}}{p_1}) - E(\frac{\mathrm{n}}{\mathrm{p}_2}) - \ldots - E(\frac{\mathrm{n}}{\mathrm{p_h}}) + o(n)} \\ &= \alpha^{n - \frac{\mathrm{n}}{\mathrm{p}_1} - \frac{\mathrm{n}}{\mathrm{p}_2} - \cdots . + o(n)} \\ &= \alpha^{(1-\sigma)n + o(n)}. \end{aligned}$$

Lemma 5: *(see [8])*

Let $n > d^2$ *then,* $T_n(a, b)$ *is not divisible by any prime number* $p > \sqrt{(n + 1)d}$ *for which* $p \equiv a$ *does not hold.*

Theorem 6: *For* $\xi \rightarrow +\infty$ *we have:*

$$\prod_{p\leq \xi\ ,p\equiv a\ mod(\mathrm{d})} p \leq (\alpha(d))^{\frac{\xi}{\mathrm{d}-1}+o(\xi)}$$

Proof;

We pose: $a+nd \leq \xi < a+(n+1)d$

By the lemma 3, $P_n(a,d)$ divisible by the prime numbers $,p \equiv a\ mod(d)$ with, $n < p < a+(n+1)d$ Then the prime numbers$,p \equiv a\ mod(d)$, with $\frac{\xi}{d} < p \leq \xi$ divides $P_n(a,d)$ (because$,n \leq \frac{\xi-a}{d} < \frac{\xi}{d}$ and $\xi < a+(n+1)d$

For $n \to +\infty$ i.e $\xi \to +\infty$, and by the lemma 1, We have:

$$\prod_{\frac{\xi}{d}\leq p\leq \xi ,p\equiv a\ mod(d)} p \leq P_n(a,d) = \alpha^{n+o(n)} \leq \alpha^{\frac{\xi}{\mathrm{d}}+\mathrm{o}(\xi)}.$$

Then for $\epsilon > 0$ $\exists \xi_0(\varepsilon)$ such that $\forall \xi \geq \xi_0(\epsilon)$ we have:

$$\prod_{\frac{\xi}{d}\leq p\leq \xi ,p\equiv a\ mod(d)} \mathrm{p} \leq \alpha^{\frac{\xi}{d}(1+\epsilon)}$$

For everything, $\xi \geq \xi_0(\epsilon)$, and $r \in N$, such that: $\frac{\xi}{d^{r+1}} < \xi_0(\epsilon) \leq \frac{\xi}{d^r}$.

We have: $\prod_{\frac{\xi}{d^2}<p\leq \frac{\xi}{d},p\equiv a\ mod(d)} p \leq \alpha^{\frac{\xi}{d^2}(1+\epsilon)}$

…………..

$$\prod_{\frac{\xi}{d^{r+1}}<p\leq \frac{\xi}{d^r},p\equiv a\ mod(d)} p \leq \alpha^{\frac{\xi}{d^{r+1}}(1+\mathrm{c})}.$$

Then:

$$\begin{aligned}\prod_{p\leq \xi ,\mathrm{p}\equiv a\ mod(\mathrm{d})} p \quad &\leq \quad \prod_{p\leq \xi_0 ,p\equiv a\ mod(d)} p \cdot \alpha^{(\frac{\xi}{d}+\frac{\xi}{d^2}+\ldots+\frac{\xi}{d^{r+1}})(1+\epsilon)} \\ &\leq \quad \prod_{p\leq \xi_0 ,P\equiv a\ mod(d)} p \cdot \alpha^{\frac{\xi}{d-1}(1+\epsilon)}.\end{aligned}$$

Then, for $\xi \geq \xi_1(\epsilon)$, we have:

$$\prod_{p\leq \xi ,p\equiv a\ mod(d)} p \leq \alpha^{\frac{\xi}{d-1}(1+2\epsilon)}.$$

So for $\xi \to +\infty$ we have:

$$\prod_{p\leq \xi ,p\equiv a\ mod(d)} p \leq (\alpha(d))^{\frac{\xi}{d-1}+\mathrm{o}(\xi)}.$$

Theorem 7: *If* $\sigma < \frac{1}{2} - \frac{d}{(d-1)(2d+1)} = \frac{2d^2-3d-1}{2(d-1)(2d+1)}$

For ξ *sufficiently large enough, there exists a prime number p of the form* $a + kd$*, such that:*

$$\xi < p \leq 2\xi.$$

Proof:

We pose:

$$\varphi_n(a,d) = \frac{T_{2n}(a,d)}{P_n(a,d)}$$

By the lemma 1 and the lemma 4 with n big enough ($n \to +\infty$)

We have:

$$\varphi_n(a,d) = a^{(1-2\sigma)n+o(n)}.$$

For, $n > d,^2$ then by the lemma 5 $\varphi_n(a,d)$ is not divisible by a prime number that does not satisfy $p \equiv a\ mod(d)$ and $p > \sqrt{(2n+1)d}$

By the lemma 3, the prime numbers $p \equiv a\ mod(d)$ with

$n < p < a + (n+1)d$, and $p > \sqrt{(2n+1)d} > \sqrt{(n+1)d}$ divides $P_n(a,d)$, while $\varphi_n(a,d)$ is not divisible by p(because: if $\varphi_n(a,d)$ is divisible by $p \Rightarrow p_{2n}(\mathrm{a},d)$ is divisible by p^2 and $p^2 > (2n+1)d$,contradiction). By the lemma 3 the prime numbers $p \equiv a\ mod(d)$ with

$$\frac{a+2nd}{1+2d} < \mathrm{p} < n, p > \sqrt{(2n+}1)d$$

Do not divide $T_{2n}(\mathrm{a},\mathrm{d})$

Then according to the lemma 2, and for $\xi \to +\infty$

We have:

$$\begin{aligned}\varphi_{\mathrm{n}}(a,d) &\leq \prod_{p\leq\sqrt{(2n+1)d}}(2n+1)d \prod_{\sqrt{(2n+1)d}<p\leq\frac{a+2nd}{1+2d},p\equiv a\ mod(d)} p \prod_{a+(n+1)d\leq p\leq a+2nd,p\equiv a\ mod\ (d)} p \\ &\leq ((2n+1)d)^{\sqrt{(2n+1)d}} \prod_{p\leq\frac{a+2nd}{1+2d},p\equiv a\ mod(d\ )} p \prod_{a+(n+1)d\leq p\leq a+2nd,p\equiv a\ mod(d)} p \\ &\leq \alpha^{\frac{a+2nd}{1+2d}\cdot\frac{1}{d-1}+o(n)} \prod_{a+(n+1)d\leq p\leq a+2nd,p\equiv a\ mod(d)} p \\ &= \alpha^{\frac{2d}{(d-1)(2d+1)}n+o(n)} \prod_{a+(n+1)d\leq p\leq a+2nd,p\equiv a\ mod(d)} p\,.\end{aligned}$$

According to (1.7), we get

$$\prod_{a+(n+1)d\leq p\leq a+2nd,p\equiv a\ mod(d)} p \geq \alpha^{(1-2\ \sigma-\frac{2d}{(d-1)(2d+1)})n+o(n)}.$$

Let n big enough, we pose:

$$a+nd\leq\ \xi<a+(n+1)d$$

Since, $a+2nd\leq 2(a+nd)\leq 2\xi$Then there is a prime number $p\equiv a\ mod(d)$ such that $\xi<\ p\ \leq 2\ \xi$

Theorem 8: *For* $\xi\geq 1$ *we have:*

$$\prod_{p\leq\ \xi,p\equiv a\ mod(d)} p\leq d\ \ \xi\ (\alpha(d))^{\frac{\xi}{\xi-1}}.$$

Proof;

We have:

$$a+(k-1)d\leq kd,\quad \forall k\in\{1,2,\dots,n\}$$

By (1.2) we get:

$$q_n(a,d)\leq\frac{\prod_{p/d}p^{(\frac{n-1}{p-1})}\prod_{k=1}^{n}(\mathrm{kd})}{n!}=d^n(\prod_{p/d}p^{\frac{1}{p-1}})^{n-1}$$

We have, the prime number $p\equiv a\ \ mod(d)$ with $n<p<a+nd$ divides $q_n(a,d)$ then:

$$\prod_{n<p<a+nd,p\equiv a\ mod(d)} p\leq d^n\ \delta^{n-1}$$

$$where\ \ \delta = \delta(d) = \frac{1}{d}\alpha(d).$$

We pose:$\{\eta\ \}$ = the smallest integer $\geq \eta$.

Let, $n_1 = \{\frac{n}{d}\}$, $n_2 = \{\frac{n}{d^2}\}$,...., $n_r = \{\frac{n}{d^r}\} = 1$, With $d^r < n \leq d^{r+1}$

We have:

$$a + n_{r+1}d \geq 1 + n_{k+1}d \geq 1 + \frac{n}{d^{r+1}} \cdot d = 1 + \frac{n}{d^r} > n_r$$

$$(k = 0,1,\dots,r,n_0 = n).$$

And also $n_{k+1} < n_k$ we find

$$\prod_{p<a+nd,p\equiv a\ mod(d)} p \leq d^{n+n_1+n_2+\dots+n_r}\ \delta^{n_1+n_2+\dots+n_r-r-1}.$$

On the other hand $n = E(\frac{n}{d^k})d^k + q, 1 \leq q < d^k, \forall k < r$

$$\Rightarrow E(\frac{n}{d^r}) \leq \frac{n-1}{d^k}.$$

$$\Rightarrow n_k \leq \frac{n-1}{d^k} + 1 = \frac{n+d^k-1}{d^k}$$

If $\frac{n}{d^k} = E\left(\frac{\text{n}}{d^k}\right) \Rightarrow n_k = \frac{n}{d^k} \leq \frac{n+d^r-1}{d^r}$;

We have:

$$\begin{aligned} n + n_1 + n_2 + \dots + n_r &\leq n + \frac{n+d-1}{d} + \frac{n+d^2-1}{d^2} + \dots + \frac{n+d^r-1}{d^r} \\ &= (n-1)(1 + \frac{1}{d} + \frac{1}{d^2} + \dots + \frac{1}{d^r}) + (r+1) \\ &< \frac{(n-1)d}{d-1} + r + 1. \end{aligned}$$

Then

$$\begin{aligned} \prod_{p<a+\text{nd},p\equiv a\ mod(d)} p &\leq d^{\frac{(n-1)d}{d-1}} \cdot d^{r+1} \cdot \delta^{\frac{(n-1)d}{d-1}} \\ &= \alpha^{\frac{(n-1)d}{d}} \cdot d^r \cdot \text{d} \\ &\leq \alpha^{\frac{(n-1)d}{d-1}} \cdot nd. \end{aligned}$$

Let $\xi \geq a$ there exists n such that $a + nd \leq \xi < a + (n+1)d$

$$\Rightarrow nd < \xi.$$

Then

$$\prod_{p \leq \xi, p \equiv a \ mod(d)} p \leq d \ \xi \ \alpha^{\frac{\xi}{d-1}}.$$

But this is verify for $\xi < a$. ▫

Main results

Circle $x^2 + y^2 =$ n! doesn't hit any lattice point any, except for: n $=$ 0, 1, 2 and 6.

Theorem 9: *For* $\xi \geq 7$ *there is a prime number* $p \equiv 3 \ mod\ 4$ *such that:*

$$\xi < p \leq 2\xi.$$

Proof:

We have:

$$7,\ 11,\ 19{,}31\ {,}59\ {,}107\ {,}211$$

$$419\ {,}827\ {,}1627\ {,}3251\ {,}6491\ {,}$$

Let $P_n = P_n(3{,}4)$ $P'_n = P_n(1{,}4)$

We pose:

$$\Phi_n = \frac{P_{2n}}{P_n P'_{E(\frac{2n}{3})}}.$$

And, $\alpha = 2 \times 4 = 8$

By (1.4), we find:

$$\begin{aligned} \Phi_n &\geq \frac{8^{2n}}{(n+1)(E(\frac{2n}{3})+1)8^{n+E(\frac{2n}{3})}} \\ &\geq (n+1)^{-1}(E(\frac{2n}{3})+1)^{-1}8^{2n-n-E(\frac{2n}{3})} \\ &\geq 3(n+1)^{-1}(2n+3)^{-1}8^{\frac{n}{3}} = 3(n+1)^{-1}(2n+3)^{-1}2^n. \end{aligned}$$

Let $n \geq 81$ then $2\sqrt{2n+1} < 4\sqrt{\mathrm{n}} < \frac{4n}{9} < \frac{8}{17}n$

By the lemma 3, we have

For n$< p < 3 + (n+1)4 = 7 + 4n$ and p/P_n

For$\frac{3+8n}{1+2\times4} = \frac{3+8n}{9} < p \leq n,$ p does not divides P_{2n}

For $E(\frac{2n}{3}) < p < \frac{3+8n}{9}$(because$\frac{1+4(E(\frac{2n}{3})+1)}{3} > \frac{1+(\frac{2n}{3})\times4}{3} = \frac{3+8n}{9}$), $p/\mathrm{P}'_{E\left(\frac{2\mathrm{n}}{3}\right)}$

For $\frac{\mathrm{n}}{2} < p < E(\frac{2n}{3})$(because $E(\frac{2}{3}) \leq \frac{2}{3} < \frac{3+4(n+1)}{1+4} = \frac{7+4n}{5}$), p/P_n.

For $\frac{3+8n}{17} < p < \frac{\mathrm{n}}{2}$, p does not divides P_{2n}

Then for $\frac{3+8n}{17} < p < 4n+7$, p does not divide$s$ Φ_n

According to Lemmas 2 and 5, we get:

$$\Phi_n \leq \prod_{p\leq\sqrt{4(2n+1)}} 4\,(2n+1) \prod_{p\leq\frac{8}{17}(n+1), p\equiv3\ mod\ 4} p \prod_{4n+7\leq p\leq8n+3, p\equiv3\ mod4} p.$$

On the other hand:

$$\prod_{p\leq\sqrt{4(2n+1)}} 4\,(2n+1) = (8n+4)^{\pi(\sqrt{8n+4}\ )},$$

$$where\ \pi(\sqrt{8\mathrm{n}+1}) = the\ \ number\ \ of\ \ prime\ \ numbers \leq \sqrt{8\mathrm{n}+4}.$$

By the theorem 8:

$$\prod_{p\leq\frac{8}{17}(n+1), p\equiv3\ mod\ 4} p \leq \frac{32}{17}(n+1)8^{\frac{8}{3\times17}(n+1)} < (n+1)2^{\frac{8n+25}{17}}.$$

Then:

$$\prod_{4n+7\leq p\leq8n+3, p\equiv3\ mod\ 4} \mathrm{p} \geq 2^{\frac{9n-25}{17}}3(n+1)^{-2}(2n+3)^{-1}(8n+4)^{-\pi(\sqrt{8n+4})}.$$

If the product is empty, then:

$$\begin{aligned} 2^{\frac{9n-25}{17}} &\leq \frac{1}{3}(n+1)^2(2n+3)(8n+1)^{\pi(\sqrt{8n+4})} \\ &= (n+1)^2(\frac{2n}{3}+1)(8n+4)^{\pi(\sqrt{8n+4})} \\ &< (n+1)^3(8n+4)^{\pi(\sqrt{8n+4})} \\ &< (2n+1)^3(8n+4)^{\pi(8n+4)} \\ &= 2^{-6}(8n+4)^{3+\pi(\sqrt{8n+4})}. \end{aligned}$$

So:

$$2^{\frac{9n+77}{17}} < (8n+4)^{3+\pi(\sqrt{8n+4})} \qquad (*)$$

For $n \geq 4$ $(*) \Rightarrow 2^{\frac{9}{17}n} < (9n)^{3+\pi(\sqrt{9n})}$

But for $\mu > 63$ we have:

$$\pi(\mu) < \frac{\mu}{3} - 3$$

Then we get

$For(n > 441)$, $2^{\frac{9}{17}n} < (9n)^{\sqrt{n}};$ because ($3\sqrt{n} > 63$)

On the other hand:

$$2^x > 2 + 2x, \ \forall x > 3$$

We find $2^{\sqrt[6]{n}} \geq 2^{E(\sqrt[6]{n})} \geq 2 + 2E(\sqrt[6]{n}) > 2\sqrt[6]{n}$ pour ($n > 729$)

$$Then\,;\ for\, n > 729,\ \ 2^{(6\sqrt[6]{n})} > 64n > 9n.$$

So

$$(*) \Rightarrow 2^{\frac{9}{17}n} < 2^{6\cdot n^{\frac{2}{3}}},$$

$$\Rightarrow 9n < 102 \cdot n^{\frac{2}{3}},$$

$$\Rightarrow n^{\frac{1}{3}} < \frac{102}{9}$$

It deduces that for $n > 1460$ then there exists $p \equiv 3 \ mod\ 4$ such that:

$$4n + 7 \leq p \leq 8n + 3$$

We pose $4n+3 \leq \xi < 4n+7$, then $\exists p \equiv 3\ mod4$ such that:

$$\xi < p \leq 2\xi.$$

Conclusion: for$\xi \geq 5850$, $\exists p \equiv 3\ mod4$, such that:

$$\xi < p \leq 2\xi.$$

Theorem10: *Let n be a positive integer the following equation:*

$$x^2 + y^2 = n!, \quad (x,y) \in Z \times Z, \qquad (I)$$

Does not admit a solution except the, $n = 0$,1,2, 6.

Proof:

If $n \geq 7$, by theorem 9, there exists a prime $p \equiv 3\ mod\ 4$ such that

$$\frac{n}{2} < p \leq n.$$

Then the equation (*I*) does not admit a solution

If $n = 6 \Rightarrow n! = 6 \times 5 \times 4 \times 3 \times 2 = (12)^2 \times 5$

The equation (*I*) $\Leftrightarrow (\frac{x}{12})^2 + (\frac{y}{12})^2 = 5.$

Since $5 = 1 + 4$ by Fermat's theorem on sums of two squares, The equation (*I*) has solutions given by

$$S = \{(12{,}24), (-24{,}12), (-12,-24), (24,-12)\}.$$

If $n = 5 \Rightarrow n! = 5 \times 4 \times 3 \times 2$

The equation (*I*) $\Leftrightarrow (\frac{x+y}{4})^2 + (\frac{x-y}{4})^2 = 15$

Since $15 = 3 + 2 \times 4$, by Fermats theorem on sums of two squares, then the equation (*I*) does not admit a solution.

If $n = 4$, $\Rightarrow n! = 4 \times 3 \times 2$.

The equation (*I*) $\Leftrightarrow$ $(\frac{x+y}{4})^2 + (\frac{x-y}{4})^2 = 3;$

By Fermat's theorem on sums of two squares, the equation does not admit a solution.

If n $= 3 \Rightarrow n! = 3 \times 2$

The equation $(I) \Leftrightarrow (\frac{x+y}{2})^2 + (\frac{x-y}{2})^2 = 3;$

By the theorem (4.1), the equation (*I*) does not admit a solution.

If n $= 2 \Rightarrow n! = 2 = 1^2 + 1^2$

by Fermat's theorem on sums of two squares, the equation (*I*) has solutions given by

$$S = \{(1,1), (-1,1), (-1,-1), (1,-1)\}.$$

If $n = 1$ or $n = 0 \Rightarrow n! = 1 = 1^2 + 0^2$

by Fermat's theorem on sums of two squares, the equation (I) has solutions given by

$$S = \{(1,0), (0,1), (-1,0), (-1,0)\}$$

4.The formula for π by number theory

Theorem 11: *Let* $N \in \mathbb{N}$ *the non-principe Diriclet character module* 4, *definied as:*

$$\chi(d) = \begin{cases} 1 \ if \quad d \equiv 1(mod4) \\ -1 \ if \quad d \equiv 3(mod4) \\ 0 \ if \quad d \ is \ even \end{cases}$$

Then, we have:

1)the total number of lattices in the circle of radius $r = \sqrt{N}$*, is equal* $\sum_{n=1}^{N} \sum_{d/n} \chi(d)$

2) The formula of π *by number theory is given by:*

$$\pi = \lim_{N \to +\infty} \frac{4}{N} \left(\sum_{n=1}^{N} \sum_{d|n} \chi(d) \right) = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots \right).$$

Proof:

Let $(N, n) \in \mathbb{N}$^2 and $(a, b) \in \mathbb{N}$^2 be such that, $a^2 + b^2 = n$. Pose $r_2(n)$ be the total number of pairs (a, b), such that $a^2 + b^2 = n$

By Jacobi's two-square theorem, we have:

$$r_2(n) = 4(d_1(n) - d_2(n)).$$

Where $d_1(n)$ is the count of divisors $d \equiv 1(mod4)$ and d_2 is the count of divisors $d \equiv 3(mod4)$. Using the definition of the symbolχ, we find

$$r_2(n) = 4\sum_{d/n}\chi(d).$$

Which implies deducing the total number of lattices, in the circle of radius $r = \sqrt{N}$ is equal:

$$\sum_{n=1}^{\mathrm{N}} r_2(n) = 4\sum_{n=1}^{N}\sum_{d|n}\chi(d).$$

Moreover:

$$\begin{aligned} 4\sum_{n=1}^{\mathrm{N}}\sum_{d|n}\chi(\mathrm{d}) &= 4\sum_{d=1}^{N}\sum_{k=1}^{\lfloor\frac{\mathrm{N}}{d}\rfloor}\chi(d) \\ &= 4\sum_{d=1}^{N}\chi(\mathrm{d})\lfloor\frac{\mathrm{N}}{d}\rfloor. \end{aligned}$$

Using the property that:

$$\frac{N}{d}-1<\lfloor\frac{N}{d}\rfloor\le\frac{N}{d}$$

Then:

$$\sum_{1\le d=4k+1\le N}\frac{N\chi(d)}{d}-\sum_{1\le \mathrm{d}=4k+1\le N}\chi(d)<\sum_{1\le d=4k+1\le N}\chi(d)\left\lfloor\frac{N}{d}\right\rfloor\le\sum_{1\le d=4k+1\le N}\frac{N}{d}\chi(d) \quad (1)$$

Similarly, we find:

$$\sum_{3\le d=4k+3\le N}\frac{\mathrm{N}}{d}\chi(d)\le\sum_{3\le d=4k+3\le N}\chi(d)\left\lfloor\frac{N}{d}\right\rfloor<\sum_{3\le d=4k+3\le N}\frac{N\chi(d)}{d}-\sum_{3\le d=4k+3\le N}\chi(d) \quad (2)$$

From (1) and (2), we find:

$$\sum_{d=1}^{N}\frac{N\chi(d)}{d}-\sum_{1\le d=4k+1\le N}\chi(d)\le\sum_{d=1}^{N}\chi(d)\lfloor\frac{N}{d}\rfloor\le\sum_{d=1}^{N}\frac{N\chi(\mathrm{d})}{d}-\sum_{3\le d=4k+3\le N}\chi(d)$$

Using Chebyshev's theorem$(1850-1851)$: there exist positive constants c_1 and c_2 such that for sufficienty large N;

$$c_1 \frac{N}{\log N} \leq \pi(N) \leq c_2 \frac{N}{\log N}$$

We find:

$$-\sum_{3 \leq d = 4k+3 \leq N} \chi(d) \leq \pi(\mathrm{N})$$

And,

$$-\pi(N) \leq -\sum_{1 \leq d = 4k+1 \leq N} \chi(d)$$

Which implies that we have:

$$N \sum_{d=1}^{N} \frac{\chi(d)}{d} - c_2 \frac{N}{\log N} \leq \sum_{d=1}^{N} \chi(d) \lfloor \frac{N}{d} \rfloor \leq N \sum_{d=1}^{N} \frac{\chi(d)}{d} + c_2 \frac{N}{\log \mathrm{N}}$$

Then

$$\sum_{d=1}^{N} \frac{\chi(d)}{d} - \frac{c_2}{\log N} \leq \frac{1}{N} \sum_{d=1}^{N} \chi(d) \lfloor \frac{N}{d} \rfloor \leq \sum_{d=1}^{N} \frac{\chi(d)}{d} + \frac{c_2}{\log N}$$

When $N \to +\infty$ we obtain:

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{d=1}^{N} \chi(d) \lfloor \frac{N}{d} \rfloor = \lim_{N \to +\infty} \sum_{d=1}^{N} \frac{\chi(d)}{d}$$

That the number of lattices in the circle (of radius, $r = \sqrt{N}$) = πN

Which implies:

$$\begin{aligned} \pi &= \lim_{N \to +\infty} \frac{4}{N} \sum_{d=1}^{N} \sum_{d \mid \mathrm{n}} \chi(d) \\ &= \lim_{N \to +\infty} \sum_{N=1}^{N} \frac{\chi(d)}{d} \\ &= 4 \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} \\ &= 4(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots). \end{aligned}$$

5.Conclusion

This article consists of two parts.

Part one: We solve equation $x^2 + y^2 = n!$, starting with the stages this type of equation has gone through from the eras of Fermat and Euler up to the present day.

Then, we introduce some technical lemmas and Theorems from reference [F]—originally in German, but translated here into English for the reader's convenience—that will be utilized to prove our results, and finally, we present our main findings:

1- **Theorem 9**: *For* $\xi \geq 7$ *there is a prime number* $p \equiv 3 \mod 4$ *such that:*

$$\xi < p \leq 2\xi.$$

2-**Theorem10:** *Let* n *be a positive integer the following equation:*

$$x^2 + y^2 = n!, \quad (x, y) \in Z \times Z, \qquad (I)$$

Does not admit a solution except the, $n = 0$,1,2, 6.

Part Two: We determine the value of π (famously known from Leibniz's series) using number theory techniques, through:

Jacobi’s Two-Square Theorem (1829)

Chebyshev’s theorem (1850 - 1851)

Finally, we can go further and study the following equation;

$$x^4 + y^4 = n!, \quad (x, y) \in Z \times Z, \qquad (II)$$

I think there are solutions for n = 0, 1, 2.

References

[1] Barbeau, E. Power Play. Mathematical Association of America, (1997)

[2] De numeris qui sunt aggregata duorum quadratorum. (Novi commentarii academiae scientiarum Petropolitanae 4 (1752/3), 1758, 3-40)

[3] Demonstratio theorematis FERMATIANI omnem numerum primum formae 4n+1 esse summam duorum quadratorum. (Novi commentarii academiae scientiarum Petropolitanae 5 (1754/5), 1760, 3-13)

[4] Euler à Goldbach, lettre CXXV

[5] L. E. Dickson, History of the Theory of Numbers, Vol. II, Ch. VI, p. 227

[6] L. E. Dickson, History of the Theory of Numbers, Vol. II, Ch. VI, p. 228

[7]Marc Chamberland, A Natural Extension of the Pythagorean Equation to Higher Dimensions, The Ramanujan Journal ,Volume 16, pages 169– 179, (2008)

[8] P. Erdös, üer die Primzahlen gewisser arithmetischer Riehen in Budapest ,11/ September 1932.

[9] Simon Stevin. l'Arithmétique de Simon Stevin de Bruges, annotated by Albert Girard, Leyde 1625, p. 622.

.