On Study $K^h$ Generalized Birecurrent Affinely Connected Space

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Abstract

In the present paper we introduce a $K^h$-generalized birecurrent space which characterized by the condition $K^i_{jkh[ℓ]} = λ_ℓ K^i_{jkh} + b^m_{ℓm} K^i_{jkh}$, $K^i_{jkh} ≠ 0$, where $λ_ℓ$ and $b^m_{ℓm}$ are non-zero covariant vector fields and covariant tensor field of second order, respectively. This space satisfies the condition of affinely connected space called $K^h$-generalized birecurrent affinely connected space.

Keywords: Finsler space; $K^h$-Generalized birecurrent space; Ricci tensor.

1. Introduction


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Let $F_n$ be an $n$-dimensional Finsler space equipped with the metric function $a F(x, y)$ satisfying the request conditions [3].

The vectors $y_i^i, y^i_j$ and the metric tensor $g_{ij}$ satisfies the following relations

(a) $y^i_{\ |k} = 0$ and (b) $g_{ij\ |k} = 0$.

The $h$ – covariant differentiation with respect to $x^k$, commute with the partial differentiation with respect to $y^j$ according to

$$\frac{\partial}{\partial y^j \Gamma^{r}_{i jk}} = 0$$

where

(1.2) \hspace{1cm} b) \hspace{0.5cm} P_{jk}^h := \left( \frac{\partial}{\partial y^j \Gamma^{r}_{i h k}} \right) y^h = \Gamma^{r}_{i h jk} y^h$

The tensor $K^i_{rkh}$ is called *Cartan’s fourth curvature tensor* which is skew-symmetric in its last two lower indices $k$ and $h$, i.e.

(1.3) \hspace{1cm} K^j_{kjh} = -K^j_{jkh}.

The associate tensor $K_{ijkh}$ of the curvature tensor $K^i_{jkh}$ is given by

(1.4) \hspace{1cm} K_{ijkh} := g_{rj} K^r_{ikh}.

The Ricci tensor $K_{jk}$ of the curvature tensor $K^i_{jkh}$ is given by

(1.5) \hspace{1cm} K^i_{jki} = K_{jk}.

The curvature tensor $K^i_{jkh}$ satisfies the following relations too

(1.6) \hspace{1cm} K^j_{kjh} y^j = H^j_{kh}$

and

(1.7) \hspace{1cm} H^j_{kjh} - K^j_{jkh} = P^i_{jk|kh} + P^r_{jk} P^i_{rkh} - h/k.

Berwald curvature tensor $H^i_{jkh}$ satisfies the relation

(1.8) \hspace{1cm} H^j_{jkh} y^j = H^j_{kh}$

and
(1.9) \[ H^i_{jkh} = \delta_j H^i_{kh}, \]

where \( H^i_{kh} \) called \( h(v) \) – torsion tensor.

Also, satisfies bianchi identity

(1.10) a) \[ H^i_{jkh} + H^i_{hjk} + H^i_{khj} = 0 \]

and it is skew- symmetric in its last two lower indices, i.e.

(1.10) b) \[ H^i_{jkh} = -H^i_{jkh}. \]

The deviation tensor \( H^i_{j} \) is positively homogeneous of degree two in \( y^i \) and satisfies

(1.11) \[ H^i_{hk} y^h = H^i_{k}, \]

(1.12) \[ H^i_{jk} = H^i_{jki}, \]

(1.13) \[ H^i_{k} = H^i_{ki}, \]

and

(1.14) \[ H = \frac{1}{n-1} H^i_{i}, \]

where \( H^i_{jk} \) and \( H \) are called \( h\)-Ricci tensor and curvature scalar, respectively. Since contraction of the indices does not affect the homogeneity in \( y^i \), hence the tensors \( H^i_{rk}, H^i_{r} \) and the scalar \( H \) are also homogeneous of degree zero, one and two in \( y^i \), respectively.

The associate tensor \( H^i_{ijkh} \) of Berwald curvature tensor \( H^i_{jkh} \) is given by

(1.15) \[ H^i_{ijkh} = g^i_{jr} H^r_{jkh}. \]

The contraction of the indices \( i \) and \( j \) in (1.10a) and by using (1.12) and the skew-symmetric property of the curvature tensor \( H^i_{jkh} \) in the last two lower indices, shows that the \( h \) – Ricci tensor satisfies

(1.16) \[ H^r_{rkh} = H^i_{hk} - H^i_{kh}. \]

The tensor \( H^i_{jh,k} \) defined by

(1.17) \[ H^i_{jh,k} := g^i_{kh} H^i_{jk}. \]

Cartan’s fourth curvature tensor \( K^i_{jkh} \) satisfies the following identity known as Bianchi identity.
\( K_{jkh}^{i} + \Gamma_{jkh}^{i} + K_{jkh}^{i} \) + \( y^{j}(\partial_{s} \Gamma_{jk}^{i})K_{rht}^{i} + (\partial_{s} \Gamma_{jh}^{i})K_{rkh}^{i} + \)
\( (\partial_{s} \Gamma_{jh}^{i})K_{rkh}^{i} \} = 0. \)

A Finsler space whose Berwald's connection parameter \( G_{jkh}^{i} \) is independent of \( y^{i} \) is called an affinely connected space (Berwald space). Thus, an affinely connected space has some properties as follows:

\[ G_{jkh}^{i} = 0 \]

and

\[ G_{jkh}^{i} = 0. \]

The connection parameters \( \Gamma_{jk}^{i} \) of Cartan and \( G_{jk}^{i} \) of Berwald coincide in an affinely connected space and they are independent of the direction arguments [3], i.e.

\[ G_{jkh}^{i} = \partial_{j}G_{kh}^{i} = 0 \]

and

\[ \partial_{j}\Gamma_{kh}^{i} = 0. \]

N. S. H. Hussein [3] introduce the \( K^{h} \)-recurrent space which characterized by the condition

\[ K_{jkh}^{i} = \lambda_{\ell}K_{jkh}^{i} + b_{\ell m}K_{jkh}^{i}, \quad K_{jkh}^{i} \neq 0, \]

where the covariant vector field \( \lambda_{\ell} \) being the recurrence vector field.

2. An \( K^{h} \) – Generalized Birecurrent Space

Let us consider a Finsler space \( F_{n} \) whose Cartan's fourth curvature tensor \( K_{jkh}^{i} \) satisfies the condition

\[ K_{jkh}^{i} = \lambda_{\ell}K_{jkh}^{i} + b_{\ell m}K_{jkh}^{i}, \quad K_{jkh}^{i} \neq 0, \]

where \( \lambda_{\ell} \) and \( b_{\ell m} \) are non-zero covariant vector field and covariant tensor field of second order, respectively.

The space satisfying the condition (2.1) will be called \( K^{h} \)-generalized birecurrent space. We shall denote it briefly by \( K^{h} \)-GBR- \( F_{n} \).

Transvecting (2.1) by \( y^{j} \), using (1.1a) and (1.6), we get

\[ H_{kh}^{i} = \lambda_{\ell}H_{kh}^{i} + b_{\ell m}H_{kh}^{i}. \]
Theorem 2.1. In $K^h$-GBR-$F_n$, the $h(\nu)$-torsion tensor $H^i_{kh}$ is generalized birecurrent.

Differentiating (2.2) partially with respect to $y^j$ and using (1.9), we get

$$\frac{\partial}{\partial t}H^i_{kh|\ell|m} = (\frac{\partial}{\partial t}\lambda t)H^i_{kh|\ell|m} + \lambda t \frac{\partial}{\partial t}(H^i_{kh|\ell|m}) + \left(\frac{\partial}{\partial t}b_{\ell|m}\right)H^i_{kh} + b_{\ell|m}H^i_{kh}. \tag{2.3}$$

Using the commutation formula exhibited by (1.2a) for $(H^i_{kh|\ell})$ and $(H^i_{kh})$ in (2.3) and using (1.9), we get

$$\left(\frac{\partial}{\partial t}(H^i_{kh|\ell})\right)_{|m} + H^r_{kh|\ell}(\frac{\partial}{\partial t}\Gamma^r_{\ell|m}) - H^r_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m}) - H^r_{kr}(\frac{\partial}{\partial t}\Gamma^r_{hm}) - H^r_{rkh}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - \frac{\partial}{\partial t}(H^r_{kh|\ell})P^r_{\ell|m} = (\frac{\partial}{\partial t}\lambda t)H^i_{kh|\ell|m} + \lambda t H^i_{kh|m} + \lambda t \left[ H^i_{rh}(\frac{\partial}{\partial t}\Gamma^r_{k|m}) - H^i_{kr}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - H^i_{rkh}P^r_{\ell|m} \right] + \left(\frac{\partial}{\partial t}b_{\ell|m}\right)H^i_{kh} + b_{\ell|m}H^i_{kh}. \tag{2.4}$$

Again applying the commutation formula exhibited by (1.2a) for $(H^i_{kh})$ in (2.4) and using (1.9), we get

$$\left(\frac{\partial}{\partial t}(H^i_{kh})\right)_{|m} + H^r_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m}) - H^r_{kr}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - H^r_{rkh}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - H^r_{rkh}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - \frac{\partial}{\partial t}(H^r_{kh})P^r_{\ell|m} = (\frac{\partial}{\partial t}\lambda t)H^i_{kh|\ell|m} + \lambda t H^i_{kh|m} + \lambda t \left[ H^i_{rh}(\frac{\partial}{\partial t}\Gamma^r_{k|m}) - H^i_{kr}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - H^i_{rkh}P^r_{\ell|m} \right] + \left(\frac{\partial}{\partial t}b_{\ell|m}\right)H^i_{kh} + b_{\ell|m}H^i_{kh}$$

which can be written as

$$\left\{\begin{align*}
H^i_{kh|\ell|m} + \left\{ H^r_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m}) - H^r_{kr}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - H^r_{rkh}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - H^r_{rkh}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - \frac{\partial}{\partial t}(H^r_{kh})P^r_{\ell|m} \right\} + \\
H^r_{kh|\ell}(\frac{\partial}{\partial t}\Gamma^r_{\ell|m}) - H^r_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m}) - H^r_{kr}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - H^r_{rkh|\ell}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - H^r_{rkh|\ell}P^r_{\ell|m} \\
- H^r_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m})P^r_{\ell|m} + H^r_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m})P^r_{\ell|m} + H^r_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m})P^r_{\ell|m} + H^r_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m})P^r_{\ell|m} \\
\lambda t H^i_{kh|m} + b_{\ell|m}H^i_{kh} + \left(\frac{\partial}{\partial t}\lambda t\right)H^i_{kh|m} + \lambda t H^i_{kh|\ell|m} + \lambda t H^i_{kh}(\frac{\partial}{\partial t}\Gamma^r_{k|m}) - \lambda t H^i_{kr}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - \lambda t H^i_{rkh}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) \\
- \lambda t H^i_{kh}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - \lambda t H^i_{kh}(\frac{\partial}{\partial t}\Gamma^r_{h|m}) - (\frac{\partial}{\partial t}b_{\ell|m})H^i_{kh}.
\end{align*}\right. \tag{2.5}$$

This shows that
\[ H^i_{jkh|\ell|m} = \lambda^i_{\ell} H^i_{jkh|m} + b^i_{\ell m} H^i_{jkh} \]

if and only if

\[
\begin{align*}
(2.6) & \quad \left\{ H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{rh}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{kr}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{rkh P^r_{f\ell}} \right\}_{|m} + H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) \\
& \quad - H^i_{rh}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{kr}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{rkh P^r_{f\ell}} - \\
& \quad H^i_{kh}(\bar{\partial}_r \Gamma^i_{r\ell}) P^r_{j\ell} + H^i_{sh}(\bar{\partial}_r \Gamma^i_{r\ell}) P^r_{j\ell} + H^i_{ks}(\bar{\partial}_r \Gamma^i_{r\ell}) P^r_{j\ell} + H^i_{skh P^r_{r\ell} P^r_{j\ell}} \\
& \quad = \left( \partial^i_j \lambda^i_{\ell} \right) H^i_{kh} + \lambda^i_{\ell} H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) - \lambda^i_{\ell} H^i_{rh}(\bar{\partial}_j \Gamma^i_{r\ell}) - \lambda^i_{\ell} H^i_{kr}(\bar{\partial}_j \Gamma^i_{r\ell}) \\
& \quad - \lambda^i_{\ell} H^i_{rkh P^r_{j\ell}} + (\bar{\partial}_j b^i_{\ell m} \lambda^i_{\ell}) H^i_{kh}.
\end{align*}
\]

Thus, we conclude

**Theorem 2.2.** In $K^h - GBR - F_n$, Berwald curvature tensor $H^i_{jkh}$ is generalized birecurrent if and only if (2.6) holds good.

Transvecting (2.5) by $g_{ip}$, using (1.1b) and (1.15), we get

\[
\begin{align*}
(2.7) & \quad H^i_{jphk|\ell|m} + g_{ip}\left[ \left( H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{rh}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{kr}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{rkh P^r_{f\ell}} \right) \right]_{|m} \\
& \quad + H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) P^r_{i\ell} + H^i_{sh}(\bar{\partial}_j \Gamma^i_{r\ell}) P^r_{i\ell} + H^i_{ks}(\bar{\partial}_j \Gamma^i_{r\ell}) P^r_{i\ell} + H^i_{skh P^r_{r\ell} P^r_{i\ell}} \\
& \quad = \left( \lambda^i_{\ell} H^i_{jphk|m} + b^i_{\ell m} H^i_{jphk} \right) + g_{ip}\left[ \left( \partial^i_j \lambda^i_{\ell} \right) H^i_{kh} \right] + \\
& \quad \lambda^i_{\ell} H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) - \lambda^i_{\ell} H^i_{rh}(\bar{\partial}_j \Gamma^i_{r\ell}) - \lambda^i_{\ell} H^i_{kr}(\bar{\partial}_j \Gamma^i_{r\ell}) - \lambda^i_{\ell} H^i_{rkh P^r_{j\ell}} + (\bar{\partial}_j b^i_{\ell m} \lambda^i_{\ell}) H^i_{kh}.
\end{align*}
\]

This shows that

\[ H^i_{jphk|\ell|m} = \lambda^i_{\ell} H^i_{jphk|m} + b^i_{\ell m} H^i_{jphk} \]

if and only if

\[
\begin{align*}
(2.8) & \quad g_{ip}\left[ \left( H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{rh}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{kr}(\bar{\partial}_j \Gamma^i_{r\ell}) - H^i_{rkh P^r_{f\ell}} \right) \right]_{|m} \\
& \quad + H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) P^r_{i\ell} + H^i_{sh}(\bar{\partial}_j \Gamma^i_{r\ell}) P^r_{i\ell} + H^i_{ks}(\bar{\partial}_j \Gamma^i_{r\ell}) P^r_{i\ell} + H^i_{skh P^r_{r\ell} P^r_{i\ell}} \\
& \quad = g_{ip}\left[ \left( \partial^i_j \lambda^i_{\ell} \right) H^i_{kh} \right] + \\
& \quad \lambda^i_{\ell} H^i_{kh}(\bar{\partial}_j \Gamma^i_{r\ell}) - \lambda^i_{\ell} H^i_{rh}(\bar{\partial}_j \Gamma^i_{r\ell}) - \lambda^i_{\ell} H^i_{kr}(\bar{\partial}_j \Gamma^i_{r\ell}) - \lambda^i_{\ell} H^i_{rkh P^r_{j\ell}} + (\bar{\partial}_j b^i_{\ell m} \lambda^i_{\ell}) H^i_{kh} -
\end{align*}
\]
\[H^i_{rkh|\ell}P^r_{jm} - H^i_{kh}(\hat{\partial}_r\Gamma^i_{\ell|})P^r_{jm} + H^i_{sh}(\hat{\partial}_r\Gamma^i_{\ell|})P^r_{jm} + H^i_{ks}(\hat{\partial}_r\Gamma^i_{\ell|})P^r_{jm} +
\]

\[H^i_{shk|\ell}P^r_{jm} = b_{ip}(\hat{\partial}_j\lambda_i)H^i_{kh|jm} + \lambda_i H^i_{kh}(\hat{\partial}_r\Gamma^i_{rm}) - \lambda_i H^i_{kr}(\hat{\partial}_r\Gamma^i_{km})\]

Thus, we conclude

**Theorem 2.3.** In \(K^h - GBR - F_n\), the associate tensor \(H_{jikh}\) of Berwald curvature tensor \(H^i_{jkh}\) is generalized birecurrent if and only if (2.8) holds good.

Contracting the indices \(i\) and \(h\) in (2.5), using (1.12) and (1.13), we get

\[(2.9)\]

\[H_{jk[\ell|m]} + \{H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - H_{rk}(\hat{\partial}_j\Gamma^p_{\ell|}) - H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - H_{rk}(\hat{\partial}_j\Gamma^p_{\ell|})\}_{jm} +\]

\[H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - H_{rk}(\hat{\partial}_j\Gamma^p_{\ell|}) - H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - H_{rk}(\hat{\partial}_j\Gamma^p_{\ell|}) -\]

\[H_{kr|\ell}P^r_{jm} - H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|})P^r_{jm} + H_s(\hat{\partial}_r\Gamma^p_{\ell|})P^r_{jm} + H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|})P^r_{jm} +\]

\[H_{sk}P^p_{r\ell}P^r_{jm} = (\partial_j\lambda_e)H_{k|jm} + \lambda_e H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - \lambda_e H_{rk}(\hat{\partial}_r\Gamma^p_{\ell|}) + (\hat{\partial}_j\lambda_{lm})H_k.\]

This shows that

\[H_{jk[\ell|m]} = \lambda_e H_{k|jm} + b_{lm}H_{jk}\]

if and only if

\[(2.10)\]

\[\{H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - H_{rk}(\hat{\partial}_j\Gamma^p_{\ell|}) - H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - H_{rk}(\hat{\partial}_j\Gamma^p_{\ell|})\}_{jm} +\]

\[H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - H_{rk}(\hat{\partial}_j\Gamma^p_{\ell|}) - H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - H_{rk}(\hat{\partial}_j\Gamma^p_{\ell|}) -\]

\[H_{kr|\ell}P^r_{jm} - H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|})P^r_{jm} + H_s(\hat{\partial}_r\Gamma^p_{\ell|})P^r_{jm} + H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|})P^r_{jm} +\]

\[H_{sk}P^p_{r\ell}P^r_{jm} = (\partial_j\lambda_e)H_{k|jm} + \lambda_e H^p_{kp}(\hat{\partial}_r\Gamma^p_{\ell|}) - \lambda_e H_{rk}(\hat{\partial}_r\Gamma^p_{\ell|}) + (\hat{\partial}_j\lambda_{lm})H_k.\]

**Theorem 2.4.** In \(K^h - GBR - F_n\), \(K\)-Ricci tensor \(H_{jk}\) in sense of Cartan is generalized birecurrent if and only if (2.10) holds good.
Contracting the indices \( i \) and \( j \) in (2.5) and using (1.16), we get

\[
(H_{hk} - H_{kh})_{(i|m} + \left\{ H_{kh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) - H_{rh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) - H_{kr}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) \right\}_{|m} \\
- H_{rkh}^i P_r^p P_{rm} \big|_{|m} + H_{kh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) - H_{rh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) - H_{kr}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) \\
- H_{kh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) P_{pm} - H_{kh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) P_{pm} + H_{kh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) P_{pm} \\
+ H_{ks}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) P_{pm} + H_{skh}^i P_{rt} P_{pm} = \lambda_t \left( H_{hk} - H_{kh} \right)_{|m} + \\
b_{\ell m} (H_{hk} - H_{kh}) + \left( \hat{\partial}_p \lambda_t \right) H_{k|m} + \lambda_t H_{rh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) - \lambda_t H_{rh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) \\
- \lambda_t H_{kr}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) \big|_m + \left( \hat{\partial}_p b_{\ell m} \right) H_{k|h}. \\
\]

This shows that

\[
(H_{hk} - H_{kh})_{(i|m} = \lambda_t (H_{hk} - H_{kh})_{|m} + b_{\ell m} (H_{hk} - H_{kh}) \\
\]

if and only if

\[
(H_{hk} - H_{kh})_{(i|m} + \left\{ H_{kh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) - H_{rh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) - H_{kr}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) \right\}_{|m} \\
+ H_{kh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) P_{pm} - H_{rh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) P_{pm} + H_{kr}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) P_{pm} \\
+ H_{ks}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) P_{pm} = b_{\ell m} (H_{hk} - H_{kh}) + \left( \hat{\partial}_p \lambda_t \right) H_{k|m} + \lambda_t H_{rh}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) \\
- \lambda_t H_{kr}^i \left( \hat{\partial}_p \Gamma_{i}^{*p} \right) \big|_m + \left( \hat{\partial}_p b_{\ell m} \right) H_{k|h}. \\
\]

Thus, we conclude

**Theorem 2.5.** In \( K^h-GBR-F_n \), the tensor \( (H_{hk} - H_{kh}) \) is generalized birecurrent if and only if (2.12) holds good.

Differentiating (1.18) covariantly with respect to \( x^m \) in the sense of Cartan and using (1.1a), we get

\[
(K_{jkh}^i)_{|m} + K_{jfhk|l}^i + K_{jkh|l}^i + y^r \left( \left( \hat{\partial}_l \Gamma_{j}^{*i} \right) K_{rhf|m} + \\
(\hat{\partial}_l \Gamma_{j}^{*i}) K_{rhf|m} + \left( \hat{\partial}_l \Gamma_{j}^{*i} \right) K_{rhf|m} \right) + y^r \left( \left( \hat{\partial}_l \Gamma_{j}^{*i} \right) \right) K_{rfh}^i \\
\]
\[ +\left( \hat{\partial}_s \Gamma^{*i}_{fj} \right)_{[m} K^s_{rkh} + \left( \hat{\partial}_s \Gamma^{*i}_{jh} \right)_{[m} K^s_{rk} \right] = 0. \]

Using (2.1) in (2.13), we get

\[ (2.14) \quad \lambda_\ell K^{i}_{jkh} + \lambda_h K^{i}_{jkh} + \lambda_k K^{i}_{jkh} + b_{\ell m} K^{i}_{jkh} + b_{hm} K^{i}_{jkh} + b_{km} K^{i}_{jkh} \]
\[ + y^r \left\{ \left( \hat{\partial}_s \Gamma^{*i}_{jk} \right) K^s_{rkh} + \left( \hat{\partial}_s \Gamma^{*i}_{j} \right) K^s_{rk} \right\} \]
\[ + y^r \left\{ \left( \hat{\partial}_s \Gamma^{*i}_{jk} \right) K^s_{rkh} + \left( \hat{\partial}_s \Gamma^{*i}_{j} \right) K^s_{rk} \right\} = 0. \]

If Cartan's fourth curvature tensor \( K^{i}_{jkh} \) is recurrent which is given by (1.23), (2.14) becomes

\[ (2.15) \quad \lambda_\ell \lambda_m K^{i}_{jkh} + \lambda_h \lambda_m K^{i}_{jk} + \lambda_k \lambda_m K^{i}_{jkh} + b_{\ell m} K^{i}_{jkh} + b_{hm} K^{i}_{jkh} + b_{km} K^{i}_{jkh} \]
\[ + \lambda_m y^r \left\{ \left( \hat{\partial}_s \Gamma^{*i}_{jk} \right) K^s_{rkh} + \left( \hat{\partial}_s \Gamma^{*i}_{j} \right) K^s_{rk} \right\} \]
\[ + y^r \left\{ \left( \hat{\partial}_s \Gamma^{*i}_{jk} \right) K^s_{rkh} + \left( \hat{\partial}_s \Gamma^{*i}_{j} \right) K^s_{rk} \right\} = 0. \]

Putting (1.18) in (2.15), we get \( K^{i}_{jkh} \)

\[ \lambda_\ell \lambda_m K^{i}_{jkh} + \lambda_h \lambda_m K^{i}_{jk} + \lambda_k \lambda_m K^{i}_{jkh} + b_{\ell m} K^{i}_{jkh} \]
\[ + b_{hm} K^{i}_{jkh} + b_{km} K^{i}_{jkh} - \lambda_m \left( K^{i}_{jkh} + K^{i}_{jkh} + K^{i}_{jkh} \right) \]
\[ + y^r \left\{ \left( \hat{\partial}_s \Gamma^{*i}_{jk} \right) K^s_{rkh} + \left( \hat{\partial}_s \Gamma^{*i}_{j} \right) K^s_{rk} \right\} = 0. \]

which can be written as

\[ (2.16) \quad b_{\ell m} K^{i}_{jkh} + b_{hm} K^{i}_{jkh} + b_{km} K^{i}_{jkh} + y^r \left\{ \left( \hat{\partial}_s \Gamma^{*i}_{jk} \right) K^s_{rkh} + \left( \hat{\partial}_s \Gamma^{*i}_{j} \right) K^s_{rk} \right\} \]
\[ + \gamma^{(\hat{\partial}_s \Gamma^{*i}_{jk})_{[m} K^s_{rkh} + \left( \hat{\partial}_s \Gamma^{*i}_{j} \right) K^s_{rk} \right\} = 0. \]

Transvecting (2.16) by \( y^j \), using (1.1a), (1.6) and (1.2b), we get

\[ b_{\ell m} H^{i}_{kh} + b_{hm} H^{i}_{kh} + b_{km} H^{i}_{kh} + p_{sk} H^{s}_{kh} + p_{sl} H^{s}_{kh} + p_{sh} H^{s}_{kh} = 0. \]

3. \( K^h \)- Generalized Birecurrent Affinely Connected Space

Let us consider an affinely connected or Berwald's space which is characterized by any one of the equivalent conditions (1.19), (1.20), (1.21) and (1.22).
Definition 3.1. The $K^h$-generalized birecurrent space is called $K^h$-generalized birecurrent affinely connected if it satisfies any one of the conditions (1.19), (1.20), (1.21) and (1.22) and denoted briefly by $K^h$-GBR-affinely connected space.

Let us consider $K^h$-GBR-affinely connected space.

If $\dot{\gamma}_j \lambda_\ell = 0, \dot{\gamma}_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.5) reduces to

\begin{equation}
H_{jk|\ell|m} = \lambda_\ell H_{jk|m} + b_{\ell m} H_{jk|\ell}.
\end{equation}

Thus, we conclude

Theorem 3.1. In $K^h$-GBR-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then Berwald curvature tensor $H_{jk|\ell}$ is generalized bireurrent.

If $\dot{\gamma}_j \lambda_\ell = 0, \dot{\gamma}_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.7) reduces to

\begin{equation}
H_{jk|m} = \lambda_\ell H_{jk|m} + b_{\ell m} H_{jk}.
\end{equation}

Thus, we conclude

Theorem 3.2. In $K^h$-GBR-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the associate tensor $H_{jk|m}$ of Berwald curvature tensor $H_{jk|\ell}$ is generalized bireurrent.

If $\dot{\gamma}_j \lambda_\ell = 0, \dot{\gamma}_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.9) reduces to

\begin{equation}
H_{jk} = \lambda_\ell H_{jk} + b_{\ell m} H_{jk}.
\end{equation}

Thus, we conclude

Theorem 3.3. In $K^h$-GBR-affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the Ricci tensor $H_{jk}$ in sense of Berwald is generalized bireurrent.

If $\dot{\gamma}_j \lambda_\ell = 0, \dot{\gamma}_j b_{\ell m} = 0$ and in view of the conditions (1.2b) and (1.22), the equation (2.11) reduces to

\begin{equation}
(H_{hk} - H_{kh})_{t|m} = \lambda_\ell (H_{hk} - H_{kh})_{m} + b_{\ell m} (H_{hk} - H_{kh}).
\end{equation}

Thus, we conclude
Theorem 3.4. In $K^h - GBR$ affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the tensor $(H_{hk} - H_{kh})$ is generalized birecurrent.

Now, transvecting (3.1) by $y^j$, using (1.1a) and (1.8), we get

\begin{equation}
H^i_{kh|\ell |m} = \lambda_{\ell} H^i_{kh|m} + b_{\ell m} H^i_{kh}.
\end{equation}

Transvecting (3.2) by $y^k$, using (1.1a) and (1.11), we get

\begin{equation}
H^i_{h|\ell |m} = \lambda_{\ell} H^i_{h|m} + b_{\ell m} H^i_{h}.
\end{equation}

Contracting the indices $i$ and $h$ in (3.2) and using (1.13), we get

\[ H^i_{k|\ell |m} = \lambda_{\ell} H^i_{k|m} + b_{\ell m} H^i_k. \]

Contracting the indices $i$ and $h$ in (3.3) and using (1.14), we get

\[ H^i_{|\ell |m} = \lambda_{\ell} H^i_{|m} + b_{\ell m} H. \]

Transvecting (3.2) by $g_{ij}$, using (1.1b) and (1.17), we get

\[ H^i_{kp,h|\ell |m} = \lambda_{\ell} H^i_{kp,h|m} + b_{\ell m} H_{kp,h}. \]

Thus, we conclude

Theorem 3.5. In $K^h - GBR$ affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the $h(v)$ torsion tensor $H^i_{kh}$, the deviation tensor $H^i_k$, the curvature vector $H_k$, the curvature scalar $H$ and the tensor $H_{kp,h}$ are all generalized birecurrent.

In view of (1.22), the equation (2.14) can be written as

\begin{equation}
\lambda_{\ell} K^i_{jkh|\ell |m} + \lambda_{h} K^i_{jk|\ell |m} + \lambda_{k} K^i_{j\ell |m} + b_{\ell m} K^i_{jk} + b_{hm} K^i_{j\ell k} + b_{km} K^i_{j\ell h} = 0.
\end{equation}

In view of (1.23), (3.4) reduces to

\[ (\lambda_{\ell} \lambda_{m} + b_{\ell m}) K^i_{jk} + (\lambda_{h} \lambda_{m} + b_{hm}) K^i_{j\ell k} + (\lambda_{k} \lambda_{m} + b_{km}) K^i_{j\ell h} = 0 \]

which can be written as

\begin{equation}
a_{\ell m} K^i_{jk} + a_{hm} K^i_{j\ell k} + a_{km} K^i_{j\ell h} = 0,
\end{equation}

where $a_{\ell m} = \lambda_{\ell} \lambda_{m} + b_{\ell m}$ is covariant tensor field of second order.
Transvecting (3.5) by $y^j$, using (1.1a) and (1.6), we get

\[(3.6) \quad a_{\ell m} H_{kh}^i + a_{hm} H_{\ell k}^i + a_{km} H_{h\ell}^i = 0.\]

Thus, we conclude

**Theorem 3.6.** In $K^h - GBR -$ affinely connected space, the identities (3.5) and (3.6) are hold good.

Contracting the indices $i$ and $\ell$ in (3.5), using (1.3) and (1.5), we get

\[(3.7) \quad a_{pm} K_{jkh}^p - a_{hm} K_{jk}^p + a_{km} K_{jh}^p = 0\]

which can be written as

\[(3.8) \quad K_{jkh}^p = \frac{(a_{nm} K_{jk}^n - a_{km} K_{jn}^p)}{a_{pm}}.\]

Thus, we conclude

**Theorem 3.7.** In $K^h - GBR -$ affinely connected space, Cartan’s fourth curvature tensor $K_{jkh}^p$ is defined by (3.8).

In view of (1.2b) and (1.22), (1.7) reduces to

\[(3.9) \quad H_{jkh}^i = K_{jkh}^i.\]

Putting (3.9) in (3.5), we get

\[(3.10) \quad a_{\ell m} H_{jkh}^i + a_{hm} H_{\ell k}^i + a_{km} H_{h\ell}^i = 0.\]

Thus, we conclude

**Theorem 3.8.** In $K^h - GBR -$ affinely connected space, Berwald curvature tensor $H_{jkh}^i$ coincide with Cartan’s fourth curvature tensor $K_{jkh}^i$ and the identity (3.10) holds good.

Contracting the indices $i$ and $\ell$ in (3.10), using (1.12), (1.10b) and the skew-symmetric property of Berwald curvature tensor $H_{jkh}^i$ in it’s last two lower indices, we get

\[a_{pm} H_{jkh}^p - a_{hm} H_{jk}^p + a_{km} H_{jm}^p = 0.\]

which can be written as

\[(3.11) \quad H_{jkh}^p = \frac{(a_{nm} H_{jk}^m - a_{km} H_{jn}^p)}{a_{pm}}.\]
Thus, we conclude

**Theorem 3.9.** In $K^h$–GBR–affinely connected space, Berwald curvature tensor is defined by (3.11).

Contracting the indices $i$ and $h$ in (3.9), using (1.12) and (1.5), we get

$$H_{jk} = K_{jk}. $$

Thus, we conclude

**Theorem 3.10.** In $K^h$–GBR–affinely connected space, Ricci tensor $H_{jk}$ in sense of Berwald coincide with Ricci tensor $K_{jk}$ of Cartan’s fourth curvature.

Transvecting (3.9) by $g_{ip}$, using (1.1b), (1.15) and (1.4), we get

$$H_{fpkh} = K_{fpkh}. $$

Thus, we conclude

**Theorem 3.11.** In $K^h$–GBR–affinely connected space, the associate curvature tensor $H_{fpkh}$ of Berwald curvature tensor coincide with the associate curvature tensor $K_{fpkh}$ of Cartan’s fourth curvature tensor.

4. Conclusion

(3.1) The $K^h$–generalized birecurrent space is called $K^h$–generalized birecurrent affinely connected if it satisfies any one of the conditions (1.19), (1.20), (1.21) and (1.22).

(3.2) In $K^h$–GBR–affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then Berwald curvature tensor $H_{ijkh}$ is generalized birecurrent.

(3.3) In $K^h$–GBR–affinely connected space, if the directional derivative of covariant vector field and covariant tensor of second order are vanish, then the $h(v)$–torsion tensor $H_{ikh},$ the deviation tensor $H_{h},$ the curvature vector $H_k,$ the curvature scalar $H$ and the tensor $H_{kp,h}$ are all generalized birecurrent.

(3.4) In $K^h$–GBR–affinely connected space, Cartan’s fourth curvature tensor $K_{ijkh}^p$ is defined by (3.8).

(3.5) In $K^h$–GBR–affinely connected space, Berwald curvature tensor $H_{ijkh}^i$ coincide with Cartan’s fourth curvature tensor $K_{ijkh}^i$ and the identity (3.10) holds good.

(3.6) In $K^h$–GBR–affinely connected space, Ricci tensor $H_{jk}$ in sense of Berwald coincide with Ricci
tensor $K_{jk}$ of Cartan's fourth curvature.

(3.7) In $K^h$–GBR–affinely connected space, the associate curvature tensor $H_{ipkh}$ of Berwald curvature tensor coincide with the associate curvature tensor $K_{ipkh}$ of Cartan's fourth curvature tensor.

5. Recommendations

Authors recommend the need for the continuing research and development in affinely connected space and its relation with other spaces.

References


