On Chromatic Number and Edge-Chromatic Number of the Ottomar Graph

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Abstract

The path graph \( P_n \), consists of the vertex set \( V = \{1,2,\ldots,n\} \) and the edge set \( E = \{(1,2),(2,3),\ldots,(n-1,n)\} \). The cycle graph \( C_n \) is the path graph, \( P_n \) with an additional edge \( \{1,n\} \). Define the Ottomar Graph, denoted by \( O_{n,m} \), to be the graph \( C_n \) \( n \in \mathbb{Z}^+ \), \( n \geq 3 \), with a vertex connected by a path \( P_2 \) to a vertex of \( C_m \), \( m \in \mathbb{Z}^+ \), \( n \geq 3 \). \( C_n \) is called the heart while \( C_m \) is called a foot (feet for plural). Note that there are \( n \) copies of \( C_m \). The chromatic number of a graph \( G \), denoted by \( \chi(G) \), is the minimum number of colors the vertices of \( G \) maybe colored such that any two adjacent vertices have different colors. The edge-chromatic number of a graph \( G \), denoted by \( \chi_e(G) \), is the minimum number of colors the edges of \( G \) maybe colored such that any two incident edges have different colors. The chromatic number and the edge-chromatic number of the ottomar graph are determined. When will the two invariants be equal or when will they be unequal? When the connecting path \( P_k \) has order greater than 2, what happens to the value of \( \chi(G) \) and \( \chi_e(G) \)? Also in the paper, the other coloring invariants are compared and investigated with chromatic number and edge-chromatic number.

Keywords: path; cycle; chromatic number; edge-chromatic number; ottomar graph; generalized ottomar graph.

1. Introduction

A pair \( G = (V,E) \) with \( E \subseteq E(V) \) is called a graph (on \( V \)). The elements of \( V \) are the vertices of \( G \), and those of \( E \) the edges of \( G \).

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The vertex set of a graph $G$ is denoted by $V_G$ and its edge set by $E_G$. Therefore $G = (V_G, E_G)$. The path graph $P_n$, consists of the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{(1,2), (2,3), \ldots, (n-1,n)\}$. The cycle graph $C_n$, is the path graph, $P_n$ with an additional edge $\{1,n\}$. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimum number of colors the vertices of $G$ maybe colored such that any two adjacent vertices have different colors. The edge-chromatic number of a graph $G$, denoted by $\chi_e(G)$, is the minimum number of colors the edges of $G$ maybe colored such that any two incident edges have different colors.

Known Result 1 [2] If $C_n$ is a cycle of order $n$, then

$$
\chi(C_n) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
3 & \text{if } n \text{ is odd}
\end{cases} \tag{1.1}
$$

and,

$$
\chi_e(C_n) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
3 & \text{if } n \text{ is odd}
\end{cases} \tag{1.2}
$$

Define the Ottomar Graph, denoted by $O_{n,m}$, to be the graph $C_n, n \in \mathbb{Z}^+, n \geq 3$, with a vertex connected by a path $P_2$ to a vertex of $C_m, m \in \mathbb{Z}^+, n \geq 3$. $C_n$ is called the heart while $C_m$ is called a foot (feet for plural). Note that there are $n$ copies of $C_m$.

![Ottomar Graph](image)

2. Identities of Chromatic Number and Edge-Chromatic Number of Ottomar Graph

**Theorem 2.1**

*For all integers $m, n \geq 3, \chi(O_{n,m}) = 3$, if:*


Case 1: $m,n$ are both odd

Case 2: $m$ is odd, $n$ is even

Case 3: $m$ is even, $n$ is odd

and

- $\chi(O_{n,m}) = 2$ if $m,n$ are even.

Proof:

- Case 1: $m$ and $n$ are both odd

If $m,n$ are odd, then by equation (1.1), $\chi(C_m) = \chi(C_n) = 3$, thus $\chi(O_{n,m}) \neq 2$. Suppose the vertices of $C_m$ and $C_n$ are colored with the same set of different colors say $a_1, a_2, a_3$. Then the path $P_2$ is attached to a vertex colored $a_i, i = 1, 2, 3$ of $C_m$ and the other end vertex is attached to a vertex colored $a_j, j = 1, 2, 3$ of $C_n$, where $a_i \neq a_j$. Hence, three is the minimum number of colors to color the vertices of $O_{n,m}$, where $m, n$ are both odd. Consequently, $\chi(O_{n,m}) = 3$.

- Case 2: $m$ is odd, $n$ is even

If $m$ is odd and $n$ is even, then by equation (1.1), $\chi(C_m) = 3$ and $\chi(C_n) = 2$, so $\chi(O_{n,m}) \neq 2$. Suppose that the vertices of $C_n$ are colored with two of the colors that also color the vertices of $C_m$, say $a_1, a_2$ for $C_n$ and $a_1, a_2, a_3$ for $C_m$. Then a path $P_2$ is attached to vertex colored $a_i, i = 1, 2, 3$ of $C_m$ and other end vertex is attached to a vertex colored $a_j, j = 1, 2, 3$ of $C_n$, where $a_i \neq a_j$. This means that the minimum number of colors to color the vertices of $O_{n,m}$, where $n$ is even and $m$ is odd is three. Thus, $\chi(O_{n,m}) = 3$.

- Case 3: $m$ is even, $n$ is odd

Proof of this case is similar to case 2.

- Case 4: $m, n$ are both even

If $m,n$ are even then by equation (1.1), $\chi(C_m) = \chi(C_n) = 2$. Suppose the vertices of $C_m$ and $C_n$ are colored with the same set of different colors, say $a_1, a_2$. Then a path $P_2$ is attached to a vertex colored $a_i, i = 1, 2$ of
$C_m$ and the other end vertex is attached to a vertex colored say $a_j, j = 1, 2$ of $C_n$, where $a_i \neq a_j$. Hence, two is the minimum number of colors to color vertices of $O_{n,m}$, where, $n, m$ are both even. Consequently, $\chi(O_{n,m}) = 2$.

**Theorem 2.2** For all integers $m, n$, $m, n \geq 3$, $\chi_e(O_{n,m}) = 3$.

**Proof:**

To prove this theorem, we consider the following cases:

- **Case 1: $m, n$ are both odd**

If $m, n$ are both odd, then by equation (1.2), $\chi_e(C_m) = \chi_e(C_n) = 3$. Thus, $\chi_e(O_{n,m}) \neq 2$ since there exist three incident edges. Suppose the edges of $C_m$ and $C_n$ are colored with the same set of different colors, say $b_i, b_j, b_k$. Then a path $P_2$ connecting $C_m$ and $C_n$ must have an edge colored with one of the colors $b_1, b_2, b_3$, say $b_i, i = 1, 2, 3$, where $b_i$ is incident to edges colored $b_j$ and $b_k$ of $C_m$ and is also incident to edges colored $b_j$ and $b_k$ of $C_n$, $b_i \neq b_j \neq b_k$. Hence, three is the minimum number of colors to color the edges of $O_{n,m}$, where $n, m$ are both odd. Consequently, $\chi_e(O_{n,m}) = 3$.

- **Case 2: $m, n$ are both even**

If $m, n$ are both even, then by equation (1.2), $\chi_e(C_m) = \chi_e(C_n) = 2$. Thus, $\chi_e(O_{n,m}) \neq 2$ since there exist three incident edges. Without a loss of generality, suppose the edges of $C_m$ and $C_n$ are colored with the same set of different colors $b_1, b_2$. Then a path $P_2$ connecting $C_m$ and $C_n$ where its edge is incident to colors $b_1, b_2$ edges of $C_m$ and $C_n$, must have an edge colored with $b_3$ such that $b_1 \neq b_2 \neq b_3$. Hence, three is the minimum number of colors to color the edges of $O_{n,m}$, where $m, n$ are both even. Consequently, $\chi_e(O_{n,m}) = 3$.

- **Case 3: $m$ is odd, $n$ is even**

If $m$ is odd, $n$ is even, then by equation (1.2), $\chi_e(C_m) = 3$ and $\chi_e(C_n) = 2$. Thus $\chi_e(O_{n,m}) \neq 2$. Without a loss of generalization, suppose that the edges of $C_n$ are colored with two of the colors that also color the edges of $C_m$, say $b_1, b_2$ colors of $C_n$ and $b_1, b_2, b_3$ colors of $C_m$. Then a path $P_2$ connecting $C_m$ and $C_n$ must have an edge colored with $b_3$ and must also be incident to edges colored $b_1$ and $b_2$ of $C_m$ and must also be incident to edges colored $b_1$ and $b_2$ of $C_n$. Hence, three is the minimum number of colors to color the edges of $O_{n,m}$, where $m$ is odd, $n$ is even. Consequently, $\chi_e(O_{n,m}) = 3$.

- **Case 4: $m$ is even, $n$ is odd**

If $m$ is even, $n$ is odd, then the proof of this case is similar to case 3.

**Corollary 2.1**
For all integers \( m, n \geq 3 \)

\[
\chi(O_{n,m}) \leq \chi_e(O_{n,m})
\]

Proof:

Note that for the cases where \( m, n \) are both odd, \( m \) is odd, \( n \) is even, and \( m \) is even, \( n \) is odd, by Theorem 2.1, \( \chi(O_{n,m}) = 3 \) and by Theorem 2.2, \( \chi_e(O_{n,m}) = 3 \). Thus, \( \chi(O_{n,m}) \leq \chi_e(O_{n,m}) \). Similarly, for cases where \( m, n \) are both even, by Theorem 2.1, \( \chi(O_{n,m}) = 2 \) and by Theorem 2.2, \( \chi_e(O_{n,m}) = 3 \). Thus, \( \chi(O_{n,m}) \leq \chi_e(O_{n,m}) \).

Therefore, in all cases, \( \chi(O_{n,m}) \leq \chi_e(O_{n,m}) \).

\[\square\]

Remark 2.1 For all integers \( k \geq 3 \), \( \chi(P_k) = \chi_e(P_k) = 2 \).

3. Generalized Ottomar Graph

Define the Generalized Ottomar Graph, \( O^k_{n,m} \), is graph \( C_n, n \in \mathbb{Z}^+, n \geq 3 \), with each vertex connected by a path \( P_k, k \in \mathbb{Z}^+, k \geq 3 \) to a vertex of \( C_m, m \in \mathbb{Z}^+, m \geq 3 \). \( C_n \) is called a heart while \( C_m \) is called a foot (feet for plural). Note that there are \( n \) copies of \( C_m \).

Theorem 3.1 For all integers \( k = 3, 4 \), \( \chi(O^k_{n,m}) = 3 \) if,

- \( m, n \) are both odd
- \( m \) is odd, \( n \) is even
- \( m \) is even, \( n \) is odd

and

\[
\chi(O^k_{n,m}) = 2, \quad \text{if} \ m, n \ \text{are both even}.
\]

Proof:

- Case 1: \( m, n \) are both odd

If \( m, n \) are both odd, then by equation (1.1), \( \chi(C_m) = \chi(C_n) = 3 \). Thus, \( \chi(O^k_{n,m}) \geq 3 \). Suppose the vertices of \( C_m \) and \( C_n \) are colored with the same set of different colors say \( a_1, a_2, a_3 \). Consider the following subcases where \( k = 3 \) (odd) and \( k = 4 \) (even):

- subcase 1.1: If \( k = 3 \)

Then a path \( P_3 \) with vertices colored with two from the same set of different colors \( a_1, a_2, a_3 \) is attached to a
vertex colored say \( a_i, i = 1,2,3 \) of \( C_m \) and the other end vertex is also connected to \( a_i \) of \( C_n \), such that the second (middle) vertex of \( P_3 \) is \( a_j, j = 1,2,3 \), where \( a_i \neq a_j \). Thus three is the minimum number of colors to color the vertices of \( O_3^{n,m} \), where \( m, n \) are both odd. Consequently, \( \chi(O_3^{n,m}) = 3 \).

- subcase 1.2: If \( k = 4 \)

Note that by Remark 2.1, \( \chi(P_4) = 2 \). Suppose further that \( P_4 \) is colored with two from the same set of different colors that color the vertices of \( C_m \) and \( C_n \), say \( a_i, a_j, i, j = 1,2,3 \). Then, the first vertex of \( P_4 \) is colored \( a_i \) of \( C_m \) and the last vertex of \( P_4 \), colored \( a_j \) is attached to \( a_j \) of \( C_n \). The other vertices of \( P_4 \) are colored \( a_i, a_j \) such that no two adjacent vertices have the same color. Thus, three is the minimum number of colors to color the vertices of \( O_4^{n,m} \), where \( m, n \) are both odd. Consequently, \( \chi(O_4^{n,m}) = 3 \).

- Case 2: \( m \) is odd, \( n \) is even

If \( m \) is odd, \( n \) is even, then by equation (1.1), \( \chi(C_m) = 3 \) and \( \chi(C_n) = 2 \). Thus, \( \chi(O_k^{n,m}) \geq 3 \). Suppose that the vertices of \( C_m \) are colored with two of the different colors that also color the vertices of \( C_m \), say \( a_1, a_2 \) colors for \( C_n \) and \( a_3, a_4, a_5 \) colors for \( C_m \). Consider the following subcases where \( k = 3 \) (odd) and \( k = 4 \) (even):

- subcase 2.1: If \( k = 3 \)

Then a path \( P_3 \) is attached to vertex colored say \( a_i, i = 1,2,3 \) of \( C_m \) and the other end vertex is also connected to a vertex colored \( a_i \) of \( C_n \), such that the second (middle) vertex of \( P_3 \) is \( a_j, j = 1,2,3 \), where \( a_i \neq a_j \). Thus, three is the minimum number of colors to color the vertices of \( O_3^{n,m} \) where \( m \) is odd and \( n \) is even. Consequently, \( \chi(O_3^{n,m}) = 3 \).

- subcase 2.2: If \( k = 4 \)

Note that by Remark 2.1, \( \chi(P_4) = 2 \). Suppose further that \( P_4 \) is colored with two from the same set of different colors that color the vertices of \( C_m \) and \( C_n \), say \( a_i, a_2 \). Then, the first vertex of \( P_4 \) is attached to a vertex colored \( a_i, i = 1,2 \) of \( C_m \) and is adjacent to a vertex colored \( a_j, j = 1,2 \), which is the second vertex of \( P_4 \), and the last vertex is then connected to a vertex colored \( a_j, j = 1,2 \) of \( C_n \), where \( a_i \neq a_j \). Note that the vertices of \( C_m \) are colored \( \{a_1, a_2, a_3\} \). Thus, three is the minimum number of colors to color the vertices of \( O_4^{n,m} \), where \( m \) is odd, \( n \) is even. Consequently, \( \chi(O_4^{n,m}) = 3 \).

- Case 3: \( m \) is even, \( n \) is odd

If \( m \) is even, \( n \) is odd, then the proof of this case is similar to case 2.

- Case 4: \( m, n \) are both even

If \( m, n \) are both even, then by equation (1.1), \( \chi(C_m) = \chi(C_n) = 2 \). Suppose the vertices of \( C_m \) and \( C_n \) are
colored with the same set of different colors say \( a_1, a_2 \). Consider the following subcases where \( k = 3 \) (odd) and \( k = 4 \) (even):

- **subcase 4.1:** If \( k = 3 \)

Then a path \( P_3 \) with vertices colored with the same set of different colors \( a_1, a_2 \) is attached to vertex colored say \( a_i, i = 1, 2, 3 \) of \( C_m \) and the other end vertex is also connected to a vertex colored \( a_i \) of \( C_n \), such that the second (middle) vertex of \( P_3 \) is \( a_j, j = 1, 2, 3 \), where \( a_i \neq a_j \). Thus, two is the minimum number of colors to color the vertices of \( O_3^{n,m} \) where \( m \) is odd and \( n \) is even. Consequently, \( \chi(O_3^{n,m}) = 2. \)

- **subcase 4.2:** If \( k = 4 \)

Note that by Remark 2.1, \( \chi(P_4) = 2. \) Suppose further that \( P_4 \) is colored with two from the same set of different colors that color the vertices of \( C_m \) and \( C_n \), say \( a_1, a_2. \) Then, the first vertex of \( P_4 \) is attached to a vertex colored \( a_i, i = 1,2, 3 \) of \( C_m \) and is adjacent to a vertex colored \( a_j, j = 1,2 \), which is the second vertex of \( P_4 \), and the second vertex is adjacent to a vertex colored \( a_i, i = 1,2 \), which is the third vertex of \( P_4 \), and the last vertex is then connected to a vertex colored \( a_j, j = 1,2 \) of \( C_n \), where \( a_i \neq a_j \). Thus, two is the minimum number of colors to color the vertices of \( O_4^{n,m} \), where \( m \) is odd, \( n \) is even. Consequently, \( \chi(O_4^{n,m}) = 2. \)  

It is easy to prove that the next corollaries hold. Proofs are similar to Theorem 3.1.

**Corollary 3.1**

*For all integers \( k \geq 3, k \) is odd,*

\[
\chi(0_{n,m}^k) = 3 \quad \text{if:}
\]

- \( m, n \) are both odd
- \( m \) is odd, \( n \) is even
- \( m \) is even, \( n \) is odd

*and*

\[
\chi(0_{n,m}^k) = 2 \quad \text{if \( m, n \) are both even.}
\]

**Corollary 3.2** For *all integers \( k \geq 2, k \) is even,* \( \chi(0_{n,m}^k) = 2. \)

**Theorem 3.2** For *all integers \( m, n \geq 3 \) and for integers \( k = 3, 4, \chi_e(0_{n,m}^k) = 3. \)
Proof:

• Case 1: \(m, n\) are both odd

If \(m, n\) are both odd, then by equation (1.2), \(\chi_e(C_m) = \chi_e(C_n) = 3\). Thus, \(\chi_e(O^{k}_{n,m}) \geq 3\). Suppose the edges of \(C_m\) and \(C_n\) are colored with the same set of different colors, say \(b_1, b_2, b_3\).

- subcase 1.1: If \(k = 3\)

Note that \(P_3\) has two edges and suppose we color its edges with two from the set of different colors that color the edges of \(C_m\) and \(C_n\), say \(b_{i}, b_{j}\). Then \(b_{i}\) color of \(P_3\) is attached to \(C_m\) and is incident to edges colored \(b_{j}\) and \(b_{k}\) of \(C_m\), while the other color of the edge of \(P_3\) say \(b_{j}\) is attached to \(C_n\) and is incident to edges colored \(b_{i}\) and \(b_{k}\) of \(C_n\), where \(b_{i} \neq b_{j} \neq b_{k}\). Hence, three is the minimum number of colors that color the edges of \(O^{3}_{n,m}\). Consequently, \(\chi_e(O^{3}_{n,m}) = 3\).

- subcase 2.1: If \(k = 4\)

Note that by Remark 3.1, \(\chi_e(P_4) = 2\) and suppose the edges of \(P_4\) are colored with two from the same set of different colors that color the edges of \(C_m\) and \(C_n\), say \(b_{i}, b_{j}\). Since \(P_4\) has three edges, suppose that \(P_4\) is colored with \(b_{i}\)'s and \(b_{j}\), \(i = 1, 2, 3\), \(j = 1, 2, 3\) such that \(b_{j}\) is the middle edge and the two \(b_{i}\)'s are first and the last edges. Then, \(b_{i}\) of \(P_4\) is connected to \(C_m\) and is incident to edges colored \(b_{j}\) and \(b_{k}\) of \(C_m\), while the other \(b_{i}\) of \(C_n\). Hence, three is the minimum number of colors that color the edges of \(O^{4}_{n,m}\). Consequently, \(\chi_e(O^{4}_{n,m}) = 3\).

• Case 2: \(m\) is odd, \(n\) is even

If \(m\) is odd, \(n\) is even, then by equation (1.2), \(\chi_e(C_m) = 3\) and \(\chi_e(C_n) = 2\). Suppose the edges of \(C_n\) are colored with two from the same set of different colors that color the edges of \(C_m\), say \(b_{1}, b_{2}\) for \(C_n\) and \(b_{1}, b_{2}, b_{3}\) for \(C_m\). By this, the entire proof follows from case 1.

• Case 3: \(m\) is even, \(n\) is odd

If \(m\) is even, \(n\) is odd, then the proof of this case is similar to case 2.

• Case 4: \(m, n\) are both even

If \(m, n\) are both even, then by equation (1.2), \(\chi_e(C_m) = \chi_e(C_n) = 2\). Suppose the edges of \(C_m\) are colored by a set of different colors say \(b_{j}, b_{k}\) and \(C_n\) is colored with the set of different colors, say \(b_{j}, b_{k}\), where \(i, j, k = 1, 2, 3\) and \(b_{1} \neq b_{j} \neq b_{k}\).

- subcase 4.1: If \(k = 3\)
Note that by Remark 2.1, $\chi_e(P_3) = 2$. Clearly, $\chi_e(O^3_{n,m}) \geq 3$ since $O^3_{n,m}$ has three edges incident to each other at the endpoints of $P_3$. Then $b_i$ color of $P_3$ is attached to $C_m$ and is incident to edges colored $b_j$ and $b_k$ of $C_m$, while the other color of the edge of $P_3$ say $b_j$ is attached to $C_n$ and is incident to edges colored $b_i$ and $b_k$ of $C_n$ and is incident to edges colored $b_i$ and $b_k$ of $C_n$, where $b_i \neq b_j \neq b_k$. Hence, three is the minimum number of colors that color the edges of $O^3_{n,m}$. Consequently, $\chi_e(O^3_{n,m}) = 3$.

- subcase 4.2: If $k = 4$

Note that $P_4$ has three edges and by Remark 3.1, $\chi_e(P_4) = 2$. Clearly, $\chi_e(O^4_{n,m}) \geq 3$ since $O^4_{n,m}$ has three edges incident to each other at the endpoints of $P_k$. Suppose that there are two $b_i's$, $i = 1, 2, 3$ and one $b_j, j = 1, 2, 3$ is the color of the middle edge, while the two $b_i's$, $i = 1, 2, 3$ are the colors of the first and the last edges. Then the first $b_i$ color is attached to $C_m$ and is incident to edges colored $b_j$ and $b_k$ colors $C_m$, while the other $b_i$ is connected to $C_n$ and is also incident to edges $b_j$ and $b_k$ colors of $C_n$. Hence, three is the minimum number of colors that color the edges of $O^4_{n,m}$. Consequently, $\chi_e(O^4_{n,m}) = 3$.

Therefore for any cases, the proof follows.

Similar arguments will hold for the last Theorem.

**Theorem 3.3** For all integers $m, n \geq 3$, and for all integers $k \geq 3$, $\chi_e(O^k_{n,m}) = 3$.

**Corollary 3.3** For all integers $m, n \geq 3$ and for any integers $k \geq 3$,

$$\chi(O^k_{n,m}) \leq \chi_e(O^k_{n,m}) \leq 3.$$

**Proof:**

Note that for cases where $m, n$ are both odd, $m$ is odd, $n$ is even and $m$ is even, $n$ is odd, by Theorem 3.1, $\chi(O^k_{n,m}) = 3$ and by Theorem 3.2, $\chi_e(O^k_{n,m}) = 3$. Thus, $\chi(O^k_{n,m}) \leq \chi_e(O^k_{n,m})$. Similarly, for cases where $m, n$ are both even, by Theorem 3.1, $\chi(O^k_{n,m}) = 2$ and by Theorem 3.2, $\chi_e(O^k_{n,m}) = 3$. Thus, $\chi(O^k_{n,m}) \leq \chi_e(O^k_{n,m})$. Therefore, in all cases, $\chi(O^k_{n,m}) \leq \chi_e(O^k_{n,m})$.

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